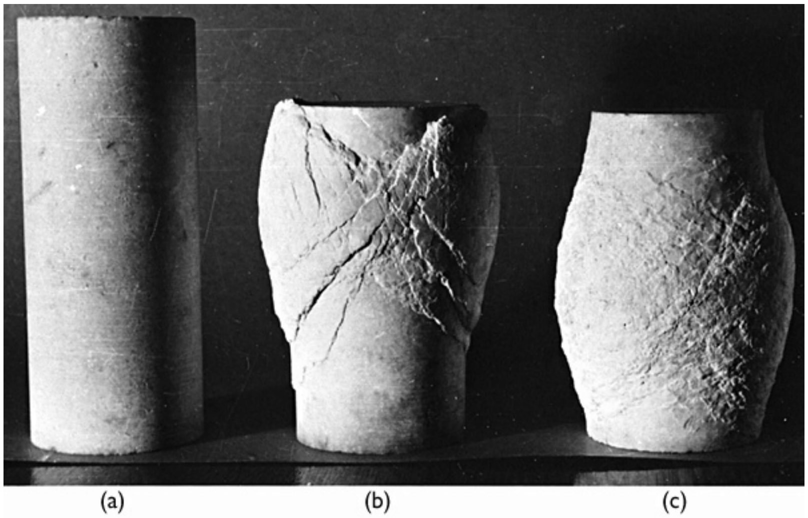
## Elasticity and Equation of Motion

- Stress and strain
- Equilibrium conditions
- Equation of motion
- Vector wave equation
- Scalar wave equations
- Applications of wave equations

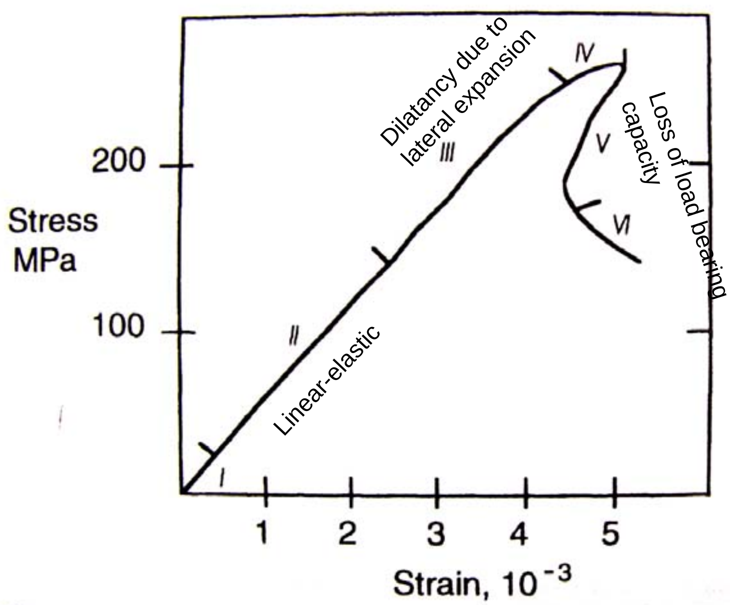
$$\rho \ddot{u}_i = f_i + \sigma_{ij,j}$$

## Assume Infinitesimal Strain Theory

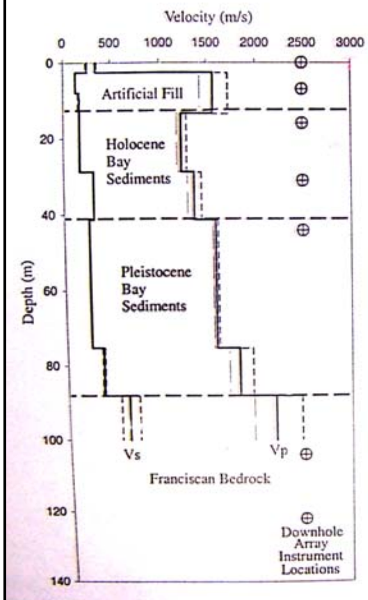
- Transient motions involve small strain where linear-elasticity holds
- Examples
  - Landers: Love wave at 150 km
    - Strain = (velocity amplitude)/phase velocity=(10cm/s)/(3.5e5cm/s)=28*$10^{-6}$
  - Landers: S-wave at 3 km
    - Strain = (150cm/s)/(3.5e5cm/s)=429*$10^{-6}$
- Notable exceptions
  - Geologic deformation
  - Near-source / fault-zone deformation
  - Transient motions in unconsolidated saturated or unsaturated materials




Typical Uniaxial Compression Test

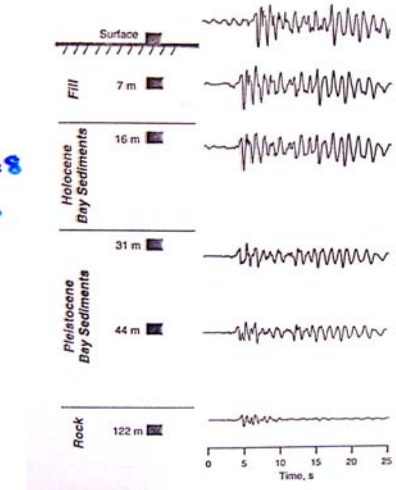


## Treasure Island



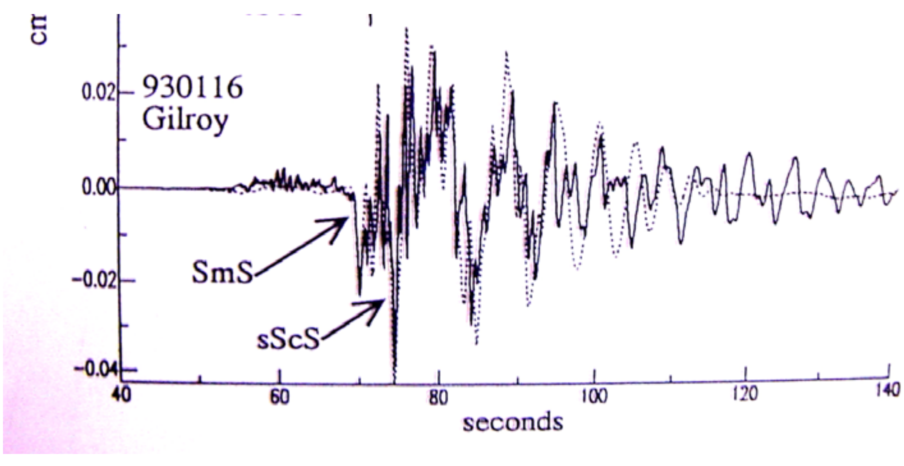

$\nu \sim 0.48$

$\alpha \sim 6\beta$

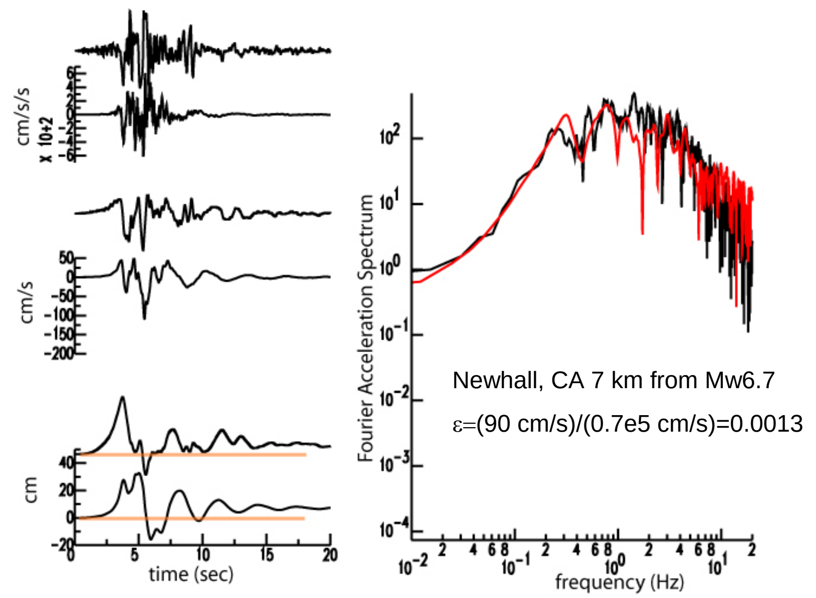



Landers: S-wave at Treasure Island

Strain = (150cm/s)/(0.25e5cm/s)=0.006

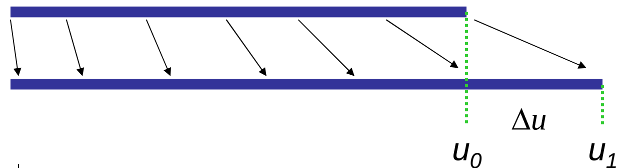


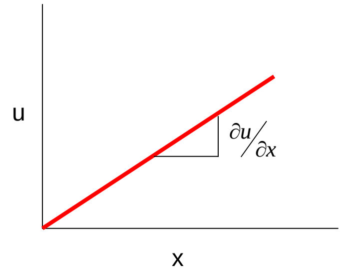
Example of Northridge modeling




# What is Strain?

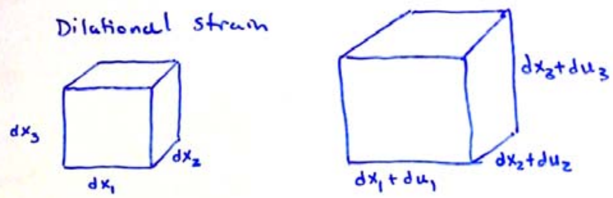
- Definition: length, volume, or angular deformation of medium due to applied force
- Linear Strain of a finite rod in tension




$$\varepsilon = \frac{u_1 - u_0}{u_0} = \frac{\partial u/\partial x \, u_0 + u_0 - u_0}{u_0} = \partial u/\partial x$$

Dilational strain




$V_0 = dx_1 dx_2 dx_3$

$$V_1 = (dx_1+du_1)(dx_2+du_2)(dx_3+du_3)$$

$$= dx_1 dx_2 dx_3 (1+\varepsilon_{11})(1+\varepsilon_{22})(1+\varepsilon_{33})$$

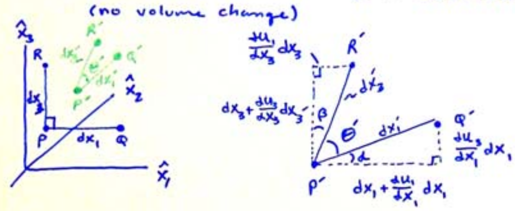
$$= V_0\left[1+(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})+\varepsilon_{11}\varepsilon_{22}+\varepsilon_{11}\varepsilon_{33}+\varepsilon_{22}\varepsilon_{33} + \varepsilon_{11}\varepsilon_{22}\varepsilon_{33}\right]$$

$$\approx V_0\left[1+(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})\right]$$

$$\Theta = \frac{\Delta V}{V_0} = \frac{V_1-V_0}{V_0} = \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} = \varepsilon_{ii}$$

Shear Strain

- measure of internal angular distortion (no volume change)




- definition

$$\varepsilon_{shear} = \frac{1}{2}\lim_{\Delta V\to 0}\left(\frac{\pi}{2}-\theta'\right)$$

if $\frac{\pi}{2}-\theta'$ is very small then

$$\varepsilon_{\substack{shear\\ \Delta V\to 0}} = \frac{1}{2}\sin\left(\frac{\pi}{2}-\theta'\right) = \frac{1}{2}\sin(\alpha+\beta)$$

$dx_1' \cos(\alpha) = dx_1 + \frac{du_1}{dx_1}dx_1$

$\cos(\alpha) = \left(1+\frac{du_1}{dx_1}\right)\frac{dx_1}{dx_1'}$

$$= \frac{1}{2}\left[\cos\alpha\sin\beta + \sin\alpha\cos\beta\right]$$

$$= \lim_{\Delta V\to 0}\frac{1}{2}\left[\left(1+\frac{du_1}{dx_1}\right)\frac{du_1}{dx_3} + \frac{du_3}{dx_1}\left(1+\frac{du_3}{dx_3}\right)\right]\frac{dx_1}{dx_1'}\frac{dx_3}{dx_3'}$$

$$\approx \frac{1}{2}\left[\frac{du_1}{dx_3}+\frac{du_3}{dx_1}\right]$$

$$\varepsilon_{ij} = \frac{1}{2}\left[\frac{du_i}{dx_j}+\frac{du_j}{dx_i}\right]$$

## Strain Tensor

$$\varepsilon_{ij} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

real & symmetric
diagonalizable
$\rightarrow$ principle strains

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$ rigid body rotation tensor

## Stress

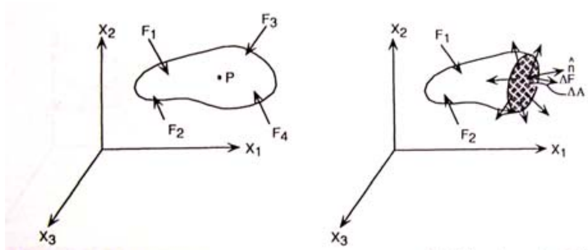
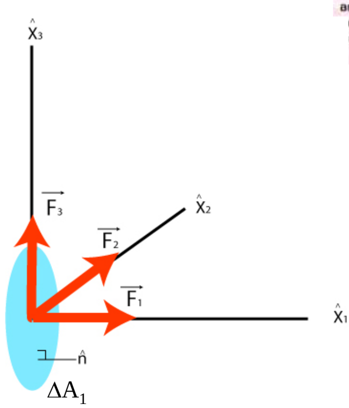



**FIGURE 2.4** (Left) A continuum acted upon by external forces. (Right) Imaginary plane with normal, **n**, passing through an internal point *P*. A portion of the medium has been removed and replaced by a distribution of forces acting on the surface, keeping the remainder of the ontinuum in equilibrium. This leads to definition of internal forces and stresses on arbitrary urfaces in the medium.




Surfaces are chosen such that:

$$\sigma_{11} = \lim_{\Delta A_1 \to 0}\left(\frac{F_1}{\Delta A_1}\right)$$ Normal stress

$$\sigma_{12} = \lim_{\Delta A_1 \to 0}\left(\frac{F_2}{\Delta A_1}\right)$$ Shear stress

$$\sigma_{13} = \lim_{\Delta A_1 \to 0}\left(\frac{F_3}{\Delta A_1}\right)$$ Shear stress

This leads to 9 stress terms, which defines the stress tensor

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

The stress acting on any arbitrary surface, or traction, can be found by the product of the stress tensor and the normal vector ($n$) of the plane

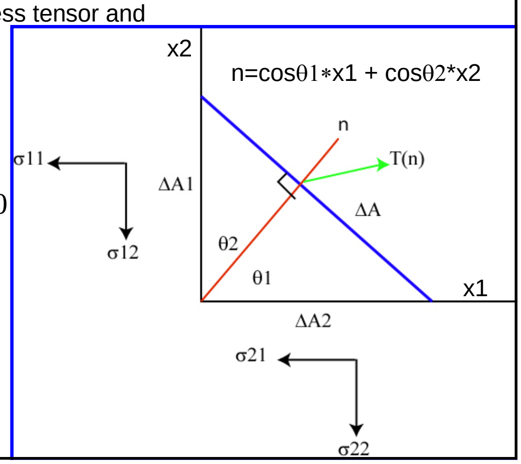
Considering the forces on all surface to be in equilibrium

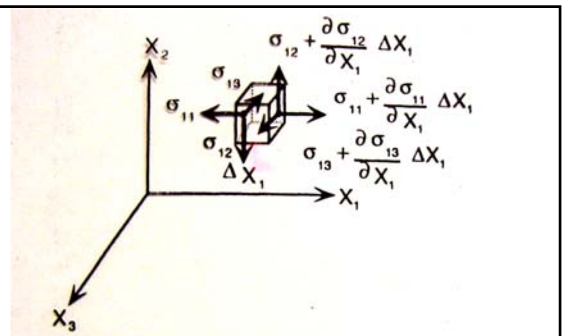
$$\Sigma F_1 = T_1 \Delta A - \sigma_{11}\Delta A_1 - \sigma_{21}\Delta A_2 = 0$$

$$\Sigma F_2 = T_2 \Delta A - \sigma_{12}\Delta A_1 - \sigma_{22}\Delta A_2 = 0$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} \cos\theta_1 \\ \cos\theta_2 \end{bmatrix}$$

$$\boxed{T(\hat{n}) = \sigma_{ji} n_j}$$




Force Balance

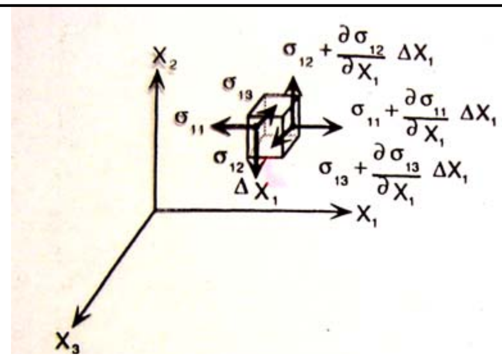
$$\Sigma F_1 = \left(\sigma_{11} + \frac{\partial\sigma_{11}}{\partial x_1}\Delta x_1 - \sigma_{11}\right)\Delta x_2 \Delta x_3 + \left(\sigma_{21} + \frac{\partial\sigma_{21}}{\partial x_2}\Delta x_2 - \sigma_{21}\right)\Delta x_1 \Delta x_3$$

$$+ \left(\sigma_{31} + \frac{\partial\sigma_{31}}{\partial x_3}\Delta x_3 - \sigma_{31}\right)\Delta x_1 \Delta x_2 ] = 0$$

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} = 0$$

generalizing for 3 components

$$* \quad \Sigma F_i = \frac{\partial\sigma_{ij}}{\partial x_i} = 0$$




Moment Balance

$$\Sigma M_{x_3} = \left[\left(\sigma_{12} + \frac{\partial\sigma_{12}}{\partial x_1}\Delta x_1\right)\frac{\Delta x_1}{2} - \sigma_{12}\left(-\frac{\Delta x_1}{2}\right)\right]\Delta x_2 \Delta x_3$$

$$-\left[\left(\sigma_{21} + \frac{\partial\sigma_{21}}{\partial x_2}\Delta x_2\right)\frac{\Delta x_2}{2} - \sigma_{21}\left(-\frac{\Delta x_2}{2}\right)\right]\Delta x_1 \Delta x_3 = 0$$

$$= \left(2\sigma_{12} + \frac{\partial\sigma_{12}}{\partial x_1}\Delta x_1\right) - \left(2\sigma_{21} + \frac{\partial\sigma_{21}}{\partial x_2}\Delta x_2\right) = 0$$

* $\sigma_{12} = \sigma_{21}$ as $\Delta x_1$ & $\Delta x_2 \to 0$

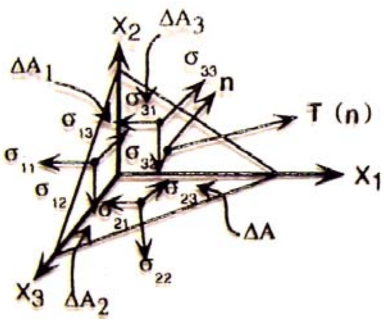
Then from moment balance the relationship

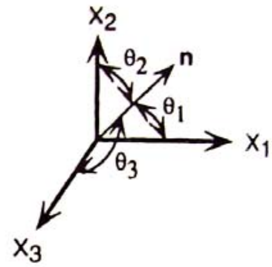
$\sigma_{ij} = \sigma_{ji}$ results

no moment → symmetric stress tensor

and

$$F = \frac{\partial \sigma_{ij}}{\partial x_j}$$

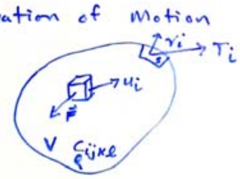



$$T(\hat{n}) = \sigma_{ji} n_j = \sigma_{ij} n_j$$

$$\vec{n} = \cos\theta_1 \cdot \hat{x}_1 + \cos\theta_2 \cdot \hat{x}_2 + \cos\theta_3 \cdot \hat{x}_3$$

Equation of Motion




Newton's 2nd Law $\vec{F} = m\vec{a}$

$$\int_V F_i \, dv + \int T_i \, ds = \int_V \rho \ddot{u}_i \, dv$$

since $T_i = \sigma_{ij} \hat{\nu}_j$

applying Gauss' Theorem $\int \vec{a} \, ds = \nabla \cdot \vec{a} \, dv$

$$\int_V F_i \, dv + \int_V \frac{\partial \sigma_{ij}}{\partial x_j} \, dv = \int_V \rho \ddot{u}_i \, dv$$

$$F_i + \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \ddot{u}_i$$ vectors are implied

$$F_1 + \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \rho \ddot{u}_1$$

## Constitutive Relationship

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$C_{ijkl}$ - 3rd order tensor with 81 elements

symmetry of $\sigma_{ij}$ & $\varepsilon_{ij}$ reduces to 36 elements

general anisotropy " " 21 elements

isotropic media " " 2 elements

where $C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

and

$$\sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$$

**TABLE 2.1** Elastic Moduli

$\mu$ Shear modulus, or rigidity. This is a measure of a material's resistance to shear.

$$\sigma_{ij} = 2\mu\varepsilon_{ij} \Rightarrow \mu = \frac{\sigma_{ij}}{2\varepsilon_{ij}}$$

Note that $\mu$ is nonnegative and has units of stress. Typical values are $2 \times 10^{11}$ dyn/cm$^2$ or 200 kbar.

$k$ Bulk modulus or incompressibility. $k$ is the material resistance to a change in volume when subject to a load, and it is defined by the ratio of an applied hydrostatic pressure to the induced fractional change in volume:

$$\sigma_{ij} = -P\delta_{ij}, \quad \frac{\Delta V}{V} = \frac{-P}{k} \Rightarrow -P = k\varepsilon_{ii} \Rightarrow \frac{-P}{\varepsilon_{ii}} = \lambda + \frac{2}{3}\mu = k$$

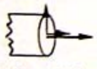
$k$ must be nonnegative, and as a material becomes more rigid, $k$ increases.

$\lambda$ Lamé's second constant. $\lambda$ has no simple physical meaning, but it greatly simplifies Hooke's law.

$E$ Young's modulus. $E$ is a measure of the ratio of uniaxial stress to strain in the same direction.


$$\sigma_{11} = E\left(\frac{\Delta L}{L}\right) = E\varepsilon_{11}; \quad \text{by Hooke's Law, } E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}$$

$\nu$ Poisson's ratio. $\nu$ is the ratio of radial to axial strain when a uniaxial stress is applied ($\sigma_{11} \neq 0$, $\sigma_{22} = \sigma_{33} = 0$).



$$\varepsilon_{22} = \varepsilon_{33}, \quad \nu = \frac{-\varepsilon_{22}}{\varepsilon_{11}} = \frac{\lambda}{2(\lambda + \mu)}$$

Poisson's ratio is dimensionless and has a maximum value of 0.5. This is true for a fluid, when $\mu = 0$ (no shear resistance). The smallest value is 0—infinite shear resistance. Most Earth materials have a Poisson ratio between 0.22 and 0.35.

**TABLE 2.2** Relationships between Elastic Moduli

| $\mu$ | $k$ | $\lambda$ | $E$ | $\nu$ |
|---|---|---|---|---|
| $\dfrac{3(k-\lambda)}{2}$ | $\lambda + \dfrac{2\mu}{3}$ | $k - \dfrac{2\mu}{3}$ | $\dfrac{9k\mu}{3k+\mu}$ | $\dfrac{\lambda}{2(\lambda+\mu)}$ |
| $\lambda\left(\dfrac{1-2\nu}{2\nu}\right)$ | $\mu\left[\dfrac{2(1+\nu)}{3(1-2\nu)}\right]$ | $\dfrac{2\mu\nu}{(1-2\nu)}$ | $2\mu(1+\nu)$ | $\dfrac{\lambda}{(3k-\lambda)}$ |
| $3k\left(\dfrac{1-2\nu}{2+2\nu}\right)$ | $\lambda\left(\dfrac{1+\nu}{3\nu}\right)$ | $3k\left(\dfrac{\nu}{1+\nu}\right)$ | $\mu\left(\dfrac{3\lambda+2\mu}{\lambda+\mu}\right)$ | $\dfrac{3k-2\mu}{2(3k+\mu)}$ |
| $\dfrac{E}{2(1+\nu)}$ | $\dfrac{E}{3(1-2\nu)}$ | $\dfrac{E\nu}{(1+\nu)(1-2\nu)}$ | $3k(1-2\nu)$ | $\dfrac{3k-E}{6k}$ |

*Table 2.3.* Representative values for density and elastic moduli in various types of rocks and various parts of the earth.

| | $\rho$ | $\kappa$ | $\mu$ | $\nu$ |
|---|---|---|---|---|
| sea water | 1.02 | 2 | 0 | 0.50 |
| young sediments | 1.9 | 3 | 0.2 | 0.47 |
| older sediments | 2.3 | 20 | 4 | 0.41 |
| granite | 2.7 | 55 | 20 | 0.34 |
| basalt | 2.9 | 75 | 35 | 0.30 |
| eclogite | 3.4 | 90 | 60 | 0.23 |
| average crust | 2.8 | 70 | 36 | 0.28 |
| average mantle | 4.5 | 350 | 200 | 0.26 |
| average outer core | 11.4 | 1030 | $<2$ | $>0.49$ |
| average inner core | 12.4 | 1400 | 130 | 0.45 |

The units of $\rho$ are $gm/cm^3$, while those of $k$ and $\mu$ are $GPa = 10^{10} dynes/cm^2$. $\nu$ has no units.

general form of equations of motion

$$\vec{F}_i + \frac{\partial}{\partial x_j}\left[\lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\right] = \rho\ddot{u}_i$$

since $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_j} + \frac{\partial u_i}{\partial x_i}\right)$ this equation yields 2nd partial of displacement & 1st partial derivatives of the elastic medium WRT space.

## Vector Wave Equation

$$\rho\ddot{u}_i = \vec{F}_i + \frac{\partial}{\partial x_j}\left[\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\right]$$

$$\rho\ddot{u}_i = \vec{F}_i + \lambda\varepsilon_{kk,j}\delta_{ij} + 2\mu\varepsilon_{ij,j}$$

$$\rho\ddot{u}_1 = \vec{F}_1 + \lambda\left[\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2 \partial x_1} + \frac{\partial^2 u_3}{\partial x_3 \partial x_1}\right] + \mu\left[\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1}\right]_{,j}$$

$$\rho\ddot{u}_1 = \vec{F}_1 + \lambda\frac{\partial\theta}{\partial x_1} + \mu\frac{\partial\theta}{\partial x_1} + \mu\left[\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right]$$

$$\rho\ddot{u}_1 = \vec{F}_1 + (\lambda + \mu)\frac{\partial\theta}{\partial x_1} + \mu\nabla^2 u_1$$

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$$\rho\ddot{u}_2 = F_2 + (\lambda+\mu)\frac{\partial\theta}{\partial x_2} + \mu\nabla^2 u_2$$

$$\rho\ddot{u}_3 = F_3 + (\lambda+\mu)\frac{\partial\theta}{\partial x_3} + \mu\nabla^2 u_3$$

Useful Vector Identities

$$\nabla(\nabla\cdot\vec{u}) = \nabla\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) = \frac{\partial\theta}{\partial x_1}\hat{x}_1 + \frac{\partial\theta}{\partial x_2}\hat{x}_2 + \frac{\partial\theta}{\partial x_3}\hat{x}_3$$

$$\nabla^2 u_1\hat{x}_1 + \nabla^2 u_2\hat{x}_2 + \nabla^2 u_3\hat{x}_3 = \nabla^2\vec{u}$$

$$\nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times\nabla\times\vec{u}$$

$$\rho\ddot{\vec{u}} = \vec{F} + (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) + \mu\nabla^2\vec{u}$$

$$\boxed{\rho\ddot{\vec{u}} = \vec{F} + (\lambda+2\mu)\nabla(\nabla\cdot\vec{u}) - \mu\nabla\times\nabla\times\vec{u}}$$

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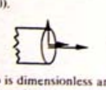

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$$\beta = \sqrt{\frac{\mu}{\rho}}$$

$$\alpha = \sqrt{\frac{\lambda+2\mu}{\rho}} = \sqrt{\frac{K+\frac{4}{3}\mu}{\rho}}$$

$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \frac{\lambda}{2(\lambda+\mu)}$ $\quad 0 \le \nu \le 0.5$

$\beta = \left[\frac{1}{2}\left(\frac{1-2\nu}{1-\nu}\right)\right]^{1/2}\alpha$ $\quad \nu = 0 \quad \beta = \frac{1}{\sqrt{2}}\alpha$

bulk earth $\quad \nu = 0.25 \quad \beta = \frac{1}{\sqrt{3}}\alpha$

Sediments $\longrightarrow$

fluid $\quad \nu = 0.5 \quad \beta = 0$

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