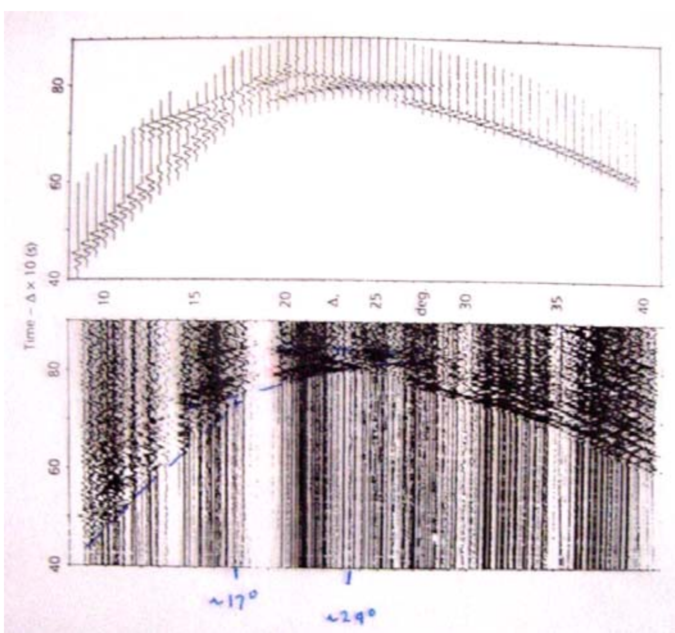
## Classical Ray Theory

- Approximation to the scalar wave equation that allows calculation of arrival times and amplitude for waves traveling in <u>smoothly</u> varying media
- What is high-frequency/short-wavelength?
- What are the capabilities?
- What are the limitations?



Suppose that $\alpha$ or $\beta$ are spatially varying functions

We have the general equation of motion

(1) $$\frac{\partial}{\partial x_j}\left( C_{ijkl}\, u_{k,l} \right) + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

considering the simplified case of a fluid

$$C_{ijkl}\, u_{k,l} = K u_{k,k}$$

and (1) can be written as

$$\frac{\partial K}{\partial x_j}\frac{\partial u_k}{\partial x_k} + K\frac{\partial}{\partial x_j}\left(\frac{\partial u_k}{\partial x_k}\right) = \rho\frac{\partial^2 u_i}{\partial t^2}$$

$$\frac{\partial K}{\partial x_j}\Theta + K\frac{\partial \Theta}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2}$$

considering volumetric motions the divergence is taken

$$\nabla\cdot\left(\frac{\partial K}{\partial x_j}\Theta + K\frac{\partial \Theta}{\partial x_j}\right) = \nabla\cdot\left(\rho\frac{\partial^2 u_i}{\partial t^2}\right)$$

$$K\frac{\partial^2 \Theta}{\partial x_i \partial x_i} - \rho\frac{\partial^2 \Theta}{\partial t^2} = \frac{\partial \rho}{\partial x_i}\frac{\partial^2 u_i}{\partial t^2} - 2\frac{\partial K}{\partial x_i}\frac{\partial \Theta}{\partial x_i} - \frac{\partial^2 K}{\partial x_i \partial x_i}\Theta$$

- does not ~~simply~~ reduce to wave equation which is easy to solve

- need to solve by
  1. Brut force - finite differences / finite elem.
  2. Ray theory approximations

## Classical Ray Theory

- Beginning with a displacement potential solution of the scalar wave equation with zero initial phase

$$\Phi = A e^{i(K_0 S(x_i) - \omega t)}$$

$K_0 = \omega / V_0$ where $V_0$ is velocity in initial reference region

$S(x_i)$ is an equation for wave fronts

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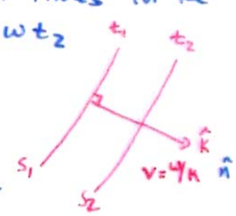
- Consider the wave front at two different times in the initial region $\xi = K_0 S_1 - \omega t_1 = K_0 S_2 - \omega t_2$

$$\frac{\Delta S}{\Delta t} = \frac{\omega}{K_0} = V_0$$

  - in another region $\dfrac{\Delta S}{\Delta t} = \dfrac{\omega}{K} = V$



- $\zeta = K_0 S(x_i) - \omega t$

  for a wave front of constant phase the total derivative is zero

  $$\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v_i \frac{\partial \zeta}{\partial x_i} = 0$$

  $$\frac{\partial \zeta}{\partial t} = -v_i \frac{\partial \zeta}{\partial x_i}$$

  $$-\omega = -v \frac{\partial \zeta}{\partial n} = -v K_0 \frac{dS}{dn}$$

  $$\frac{dS}{dn} = \frac{\omega}{v K_0} = v_0/v$$

* $$\boxed{\left(\frac{dS}{dn}\right)^2 = \left(\frac{dS}{dx_1}\right)^2 + \left(\frac{dS}{dx_2}\right)^2 + \left(\frac{dS}{dx_3}\right)^2 = \left(\frac{v_0}{v_i}\right)^2}$$ Eikonal EQn

$v = v(\vec{x})$

eqn 3.9
note $W(\vec{x})$ in book
is equivalent to
$S(\vec{x})$ in notes

image equation, which tracks trajectory changes of a wavefront passing through media of varying velocity

- $n = v_0/v$ refractive index
- velocities are defined normal to wavefront

- Applying the Helmholtz equation $(\nabla^2 + k^2)\Phi = 0$ to the specific form of $\Phi$

$k = \omega/v = k_0 \, v_0/v$

$k_0 = \omega/v_0$

$$(\nabla^2 + (k_0 \tfrac{v_0}{v})^2)\, e^{i(k_0 S(x_i) - \omega t)} = \nabla\cdot\nabla(e^{i(k_0 S(x_i) - \omega t)}) + k_0 \tfrac{v_0}{v}\, e^{i(k_0 S(x_i) - \omega t)} = 0$$

$$\nabla\left[\left(ik_0 \frac{dS(x_i)}{dx_1},\ ik_0 \frac{dS(x_i)}{dx_2},\ ik_0 \frac{dS(x_i)}{dx_2}\right) e^{i()}\right] + (k_0 \tfrac{v_0}{v})^2 e^{i()} = 0$$

$$ik_0 \frac{d^2 S(x_i)}{dx_i dx_i} - k_0^2 \frac{dS(x_i)}{dx_i}\frac{dS(x_i)}{dx_i} + (k_0 \tfrac{v_0}{v})^2 = 0$$

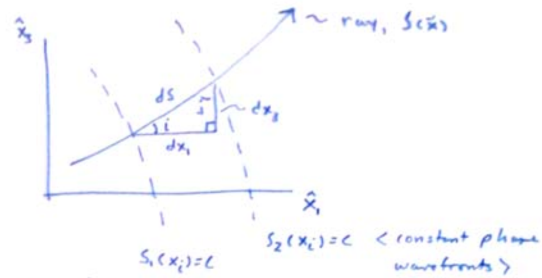
reduces to Eikonal equation if

$$\boxed{\left| ik_0 \frac{d^2 S(x_i)}{dx_i dx_i} \right| \ll \left| k_0^2 \frac{dS(x_i)}{dx_i}\frac{dS(x_i)}{dx_i} \right|}$$

condition of validity

which corresponds to high frequency & short wavelength with respect to change in seismic velocity in the direction of the ray

Consider the geometry




$\nu_1 = \frac{dx_1}{ds} = \cos i$

$\nu_2 = \frac{dx_3}{ds} = \cos j$

(*) $\left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 + \left(\frac{dx_3}{ds}\right)^2 = \nu_i \nu_i \equiv 1$

since the gradient $S(\vec{x})$ is normal (*) & the Eikonal equation are proportional &

(**) $\left(a \frac{dS(\vec{x})}{dx_1}\right)^2 + \left(a \frac{dS(\vec{x})}{dx_2}\right)^2 + \left(a \frac{dS(\vec{x})}{dx_3}\right)^2 = 1$

if $a = v_i/v_0$ then (**) is the Eikonal Equation

This gives the normal equations

$$\frac{dS(\vec{x})}{dx_1} = n\frac{dx_1}{ds}$$

$$\frac{dS(\vec{x})}{dx_2} = n\frac{dx_2}{ds} \qquad \Rightarrow \qquad \frac{dS(\vec{x})}{dx_i} = n\frac{dx_i}{ds}$$

$$\frac{dS(\vec{x})}{dx_3} = n\frac{dx_3}{ds} \qquad n = \frac{V_0}{V}$$

index of refraction

Back to condition of validity

Evaluating the change of the normal equations along the ray path

$$\frac{d}{ds}\left(\frac{dS(\vec{x})}{dx_i}\right) = \frac{d}{ds}\left(n\frac{dx_i}{ds}\right)$$

since $\nu_1 = \cos i = \frac{dx_1}{ds}$, etc. from previous diagram

$$\frac{d}{ds} = \frac{d\nu_i}{dx_i} = \frac{d}{dx_i}\frac{dx_i}{ds}$$

the left hand side becomes

$$\frac{d}{ds}\left(\frac{dS(\vec{x})}{dx_i}\right) = \frac{d}{dx_i}\left[\frac{dx_i}{ds}\frac{dS(\vec{x})}{dx_i}\right] = \frac{d}{dx_i}\left[n\left(\frac{dx_i}{ds}\right)^2\right]$$

$$= \frac{dn}{dx_i}$$

$\frac{dn}{dx_i} = \frac{d}{ds}\left(n\frac{dx_i}{ds}\right)$ is the ray path equation

given 1) initial direction of wave, $\left.\frac{dx_i}{ds}\right|_{x=s_0}$ in $V_0$

the wave can be tracked in material of changing $V$

## Example

Suppose $V(\vec{x})$ is a function of $x_3$, $V(x_3)$

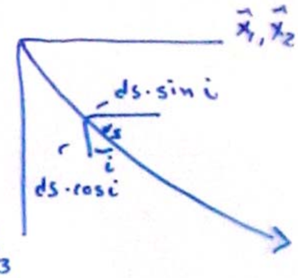
and $n = n(x_3)$ then $\frac{dn}{dx_1} = \frac{dn}{dx_2} = 0$

implying $n \frac{dx_1}{ds} = \text{constant}$

$n \frac{dx_2}{ds} = \text{constant}$

&

$$\frac{dn}{dx_3} = \frac{d}{ds}\left(n \frac{dx_3}{ds}\right)$$




$$n \frac{dx_1}{ds} = \frac{V_0}{V} \sin i = \text{constant}$$

*** $\frac{\sin i}{V} = p$ for a given ray there is a constant horizontal slowness or ray parameter

This is Snell's Law

Now for the vertical behavior

$$\frac{dn}{dx_3} = \frac{d}{ds}\left(n \frac{dx_3}{ds}\right) = \frac{d}{ds}(n \cos i)$$

$$= -n \sin i \frac{di}{ds} + \cos i \frac{dn}{dx_3} \frac{dx_3}{ds}$$

$$= -n \sin i \frac{di}{ds} + \cos^2 i \frac{dn}{dx_3}$$

$$\frac{dn}{dx_3}(\sin^2 i) = -n \sin i \frac{di}{ds} \qquad \frac{di}{ds} = -\frac{\sin i}{n} \frac{dn}{dx_3}$$

$$\frac{di}{ds} = \frac{-\sin i}{n}\frac{dn}{dx_3} = \frac{-\sin i \cdot v}{V_0} V_0 \left(\frac{d(1/v)}{dx_3}\right)$$

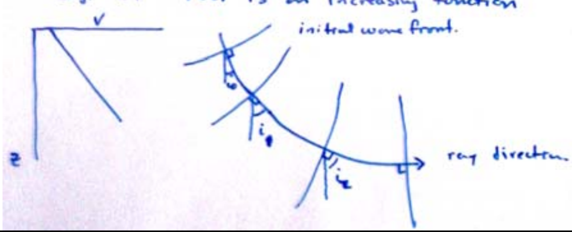
$$= -\sin i \cdot v \left(\frac{-1}{v^2}\frac{dv}{dx_3}\right)$$

$$= \frac{\sin i}{v}\frac{dv}{dx_3} = p\frac{dv}{dx_3}$$

thus the change in ray angle, $i$, depends on both the ray parameter, $p$, & the vertical velocity gradient

This is a complete description of Snell's Law, which is more commonly written $p = \frac{\sin i_0}{V_0} = \frac{\sin i_1}{V_1} = \frac{\sin i_2}{V_2}$

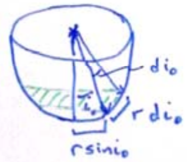
e.g. if $V(z)$ is an increasing function




Wave Amplitude in Classical Ray Theory

Energy is thought to propagate as a beam (bundle) of nearby rays.

As rays converge or diverge the arriving energy increases or decreases.

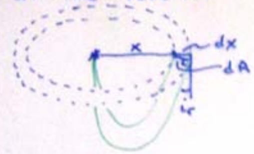


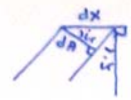

- energy per unit time is constant
- energy per unit surface area will decrease as wavefront expands
- $E = K/2\pi r^2 = \frac{\text{initial energy}}{\text{area of hemisphere}}$

Energy contained in a circular ring

$$E = \frac{K}{2\pi r^2}(2\pi r \sin i_0)(r\,di_0) = K \sin i_0\, di_0$$

at the receiver




$dA = dx \cos i_r$

At the receiver distance energy is spread

over $2\pi x \, dx \cos i_r$

then $E(x) = \dfrac{K \sin i_0 \, di_0}{2\pi x \, dx \cos i_r}$ — if $\dfrac{di_0}{dx}$ increases amplitudes increase

since $p = \dfrac{\sin i}{v_0} = \dfrac{dT}{dx}$ $\quad i_0 = \sin^{-1}\left(v_0 \dfrac{dT}{dx}\right)$

$$\frac{di_0}{dx} = \frac{d}{dx}\left(\sin^{-1}\left(v_0 \frac{dT}{dx}\right)\right)$$

$$= \frac{v_0}{\cos i_0} \frac{d^2T}{dx^2}$$

&

$$E(x) = \left(\frac{K}{2\pi}\right) v_0 \frac{\tan i_0}{x \cos i_r} \frac{d^2T}{dx^2}$$

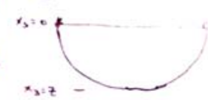
- Some interesting aspects of $\dfrac{dn}{dx_i} = \dfrac{d}{ds}\left(n \dfrac{dx_i}{ds}\right)$

$$\sin i = \frac{dx_1}{ds} = vp$$

$$\cos i = \frac{dx_3}{ds} = [1 - v^2p^2]^{1/2}$$

$$dx_1 = vp\, ds = \frac{vp}{[1 - v^2p^2]^{1/2}} dx_3$$

$$X(p) = 2\int_0^z \frac{p\, dx_3}{\left[\frac{1}{v^2} - p^2\right]^{1/2}}$$




change in traveltime = $\dfrac{\text{distance over ray path}}{\text{velocity}}$

$$dT = \frac{ds}{v}$$

$$T = \int \frac{ds}{v} = 2\int_0^z \frac{dx_3}{\cos i \, v(x_3)} = 2\int_0^z \frac{dx_3}{v(x_3)\, \eta(p)}$$

$$T(p) = 2\int_0^z \frac{dx_3}{v^2(x_3)\left[\frac{1}{v^2(x_3)} - p^2\right]^{1/2}}$$

$$= 2\int_0^z \frac{p^2}{[\ ]^{1/2}} + \frac{\frac{1}{v^2} - p^2}{[\ ]^{1/2}} \, dx_3$$

$$= pX(p) + 2\int_0^z \left[\frac{1}{v^2} - p^2\right]^{1/2} dx_3$$

– traveltime is seperable into horizontal & vertical terms

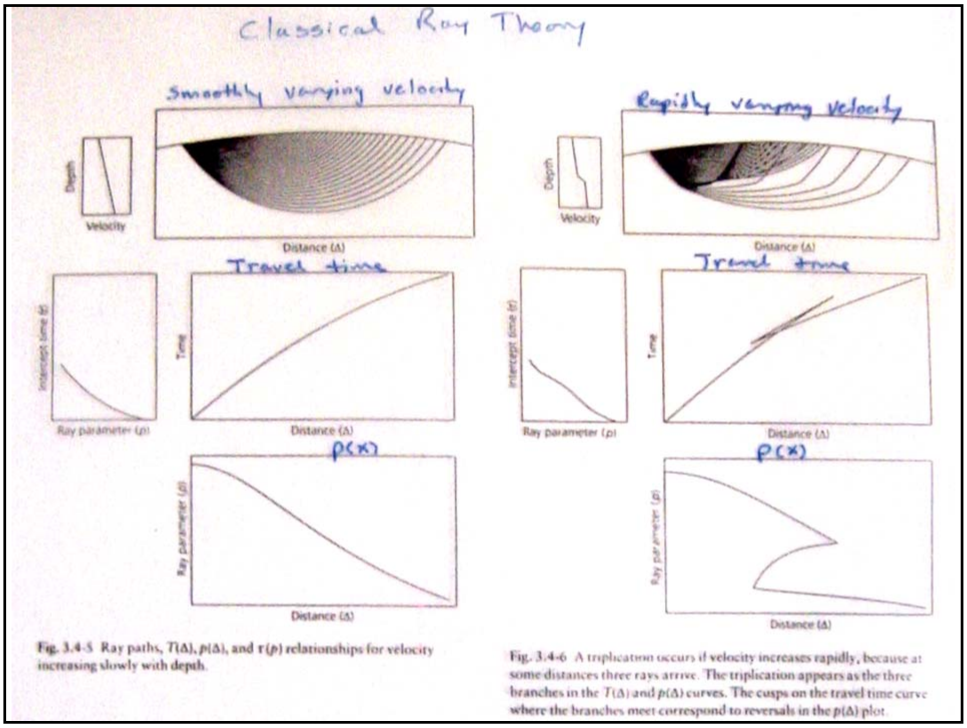



Fig. 3.4-5 Ray paths, $T(\Delta)$, $p(\Delta)$, and $\tau(p)$ relationships for velocity increasing slowly with depth.

Fig. 3.4-6 A triplication occurs if velocity increases rapidly, because at some distances three rays arrive. The triplication appears as the three branches in the $T(\Delta)$ and $p(\Delta)$ curves. The cusps on the travel time curve where the branches meet correspond to reversals in the $p(\Delta)$ plot.

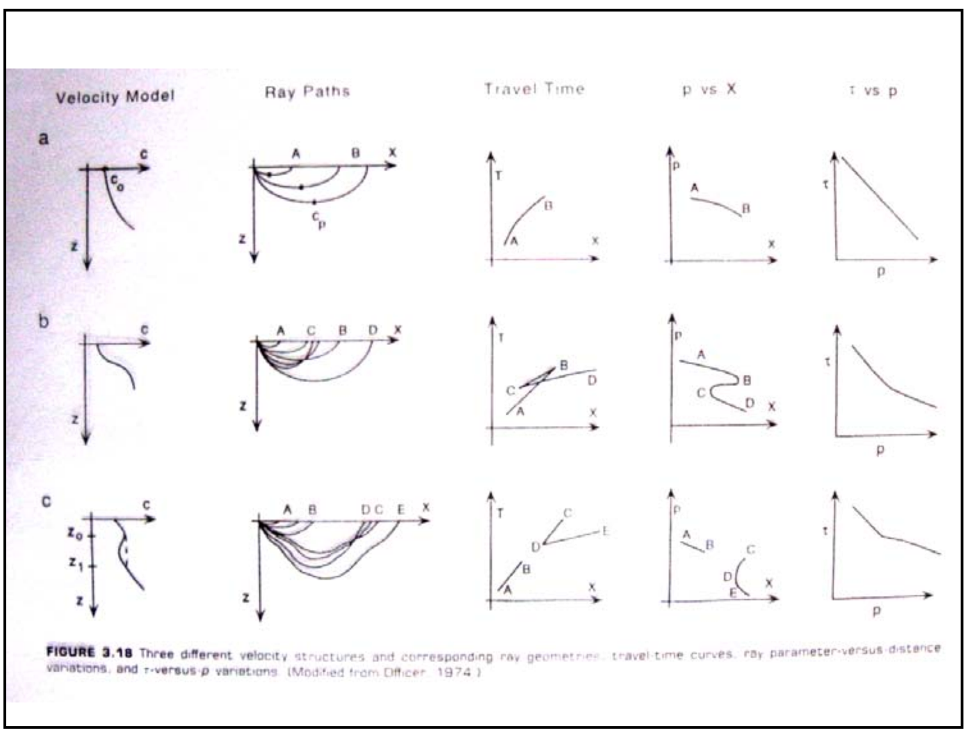
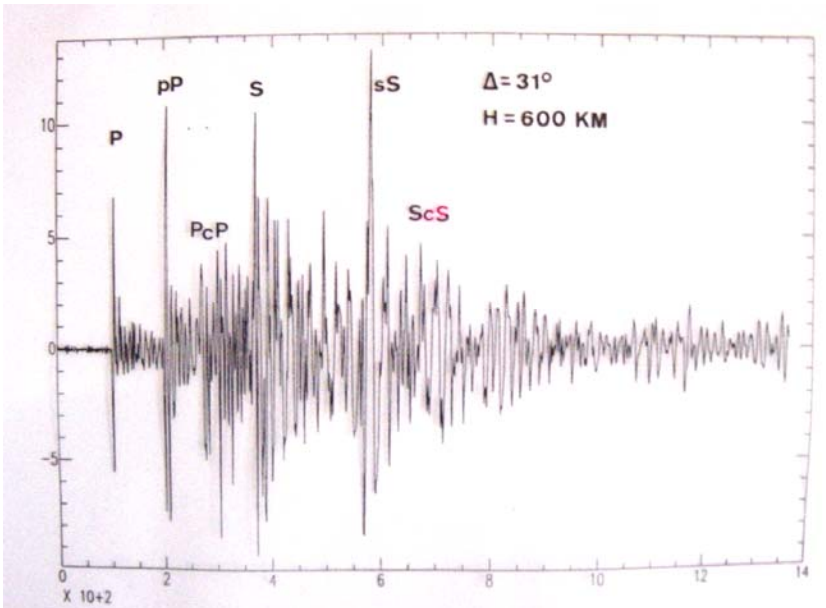
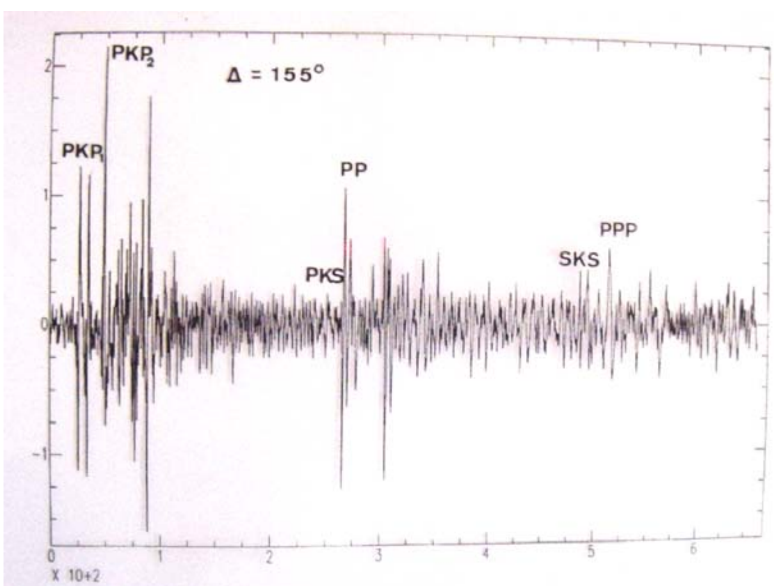



**FIGURE 3.18** Three different velocity structures and corresponding ray geometries, travel-time curves, ray parameter-versus-distance variations, and $\tau$-versus-$p$ variations. (Modified from Officer, 1974.)

## Examples



P
pP
S
sS
PcP
ScS
Δ= 31°
H = 600 KM
10
5
0
-5
0
2
4
6
8
10
12
14
X 10+2

PKP2
PKP1
Δ = 155°
PP
PKS
SKS
PPP
2
1
0
-1
0
1
2
3
4
5
6
X 10+2

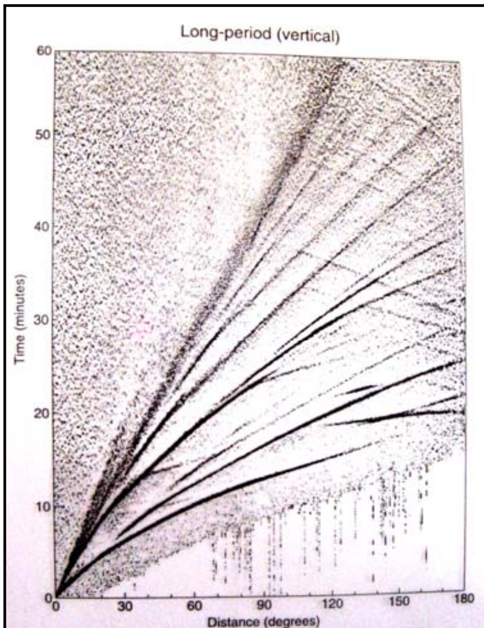



Fig. 4.20. A stack of long-period (>10 s), vertical component data from the global networks between 1988 to 1994. See Figure 4.23 for a key to the phase names. (From Astiz et al., 1996.)

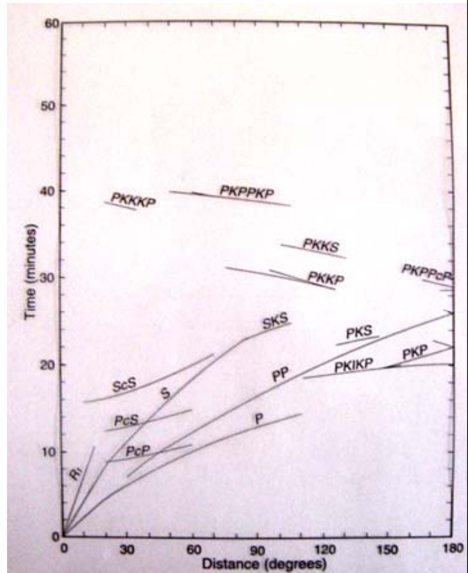



Fig. 4.19. A key to the phases visible in the short-period stack plotted in Fig. 4.18. Travel time curves are calculated using the IASP91 velocity model (Kennett and Engdahl, 1991). (From Astiz et al., 1996.)

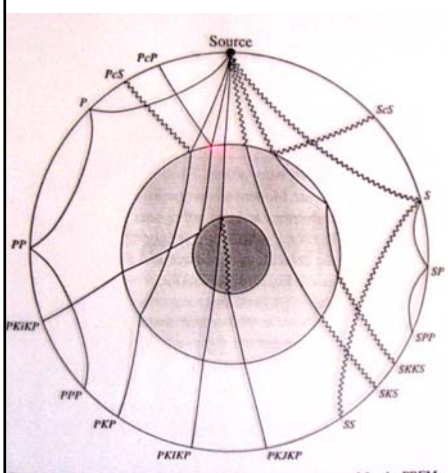
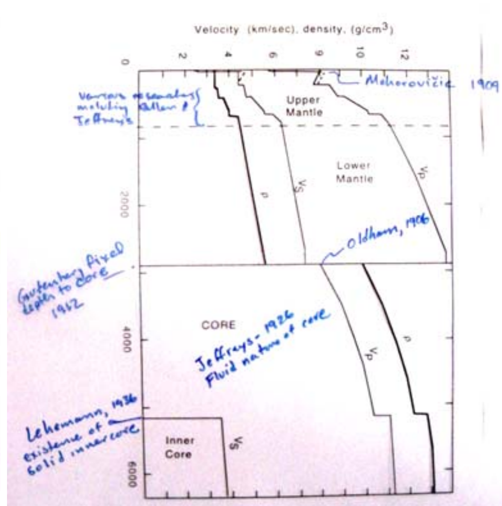
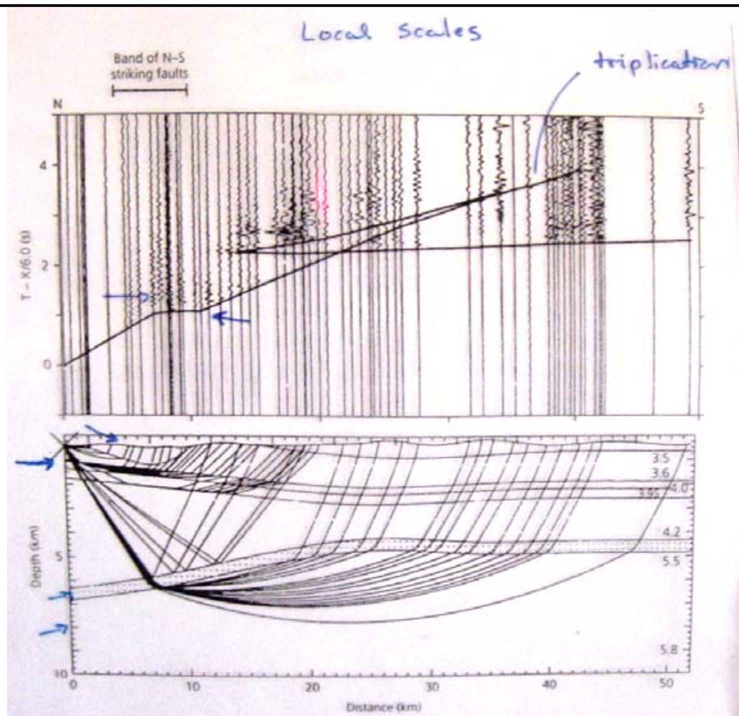
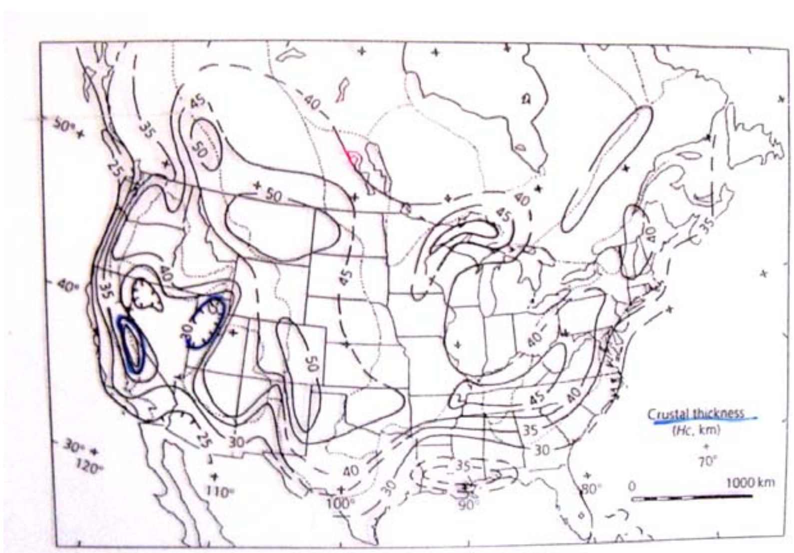



Fig. 4.14. Global seismic ray paths and phase names, computed for the PREM velocity model. *P*-waves are shown as solid lines, *S*-waves as wiggly lines. The different shades indicate the inner core, the outer core, and the mantle.

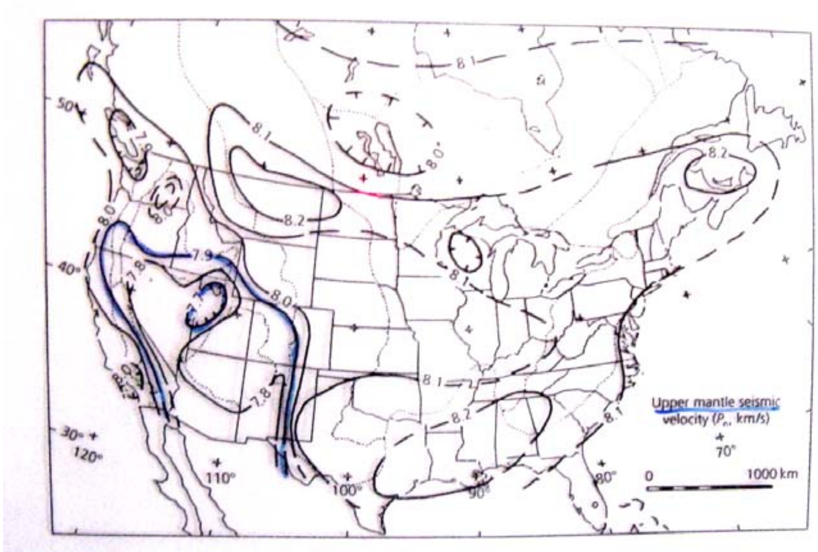





# Crustal Thickness

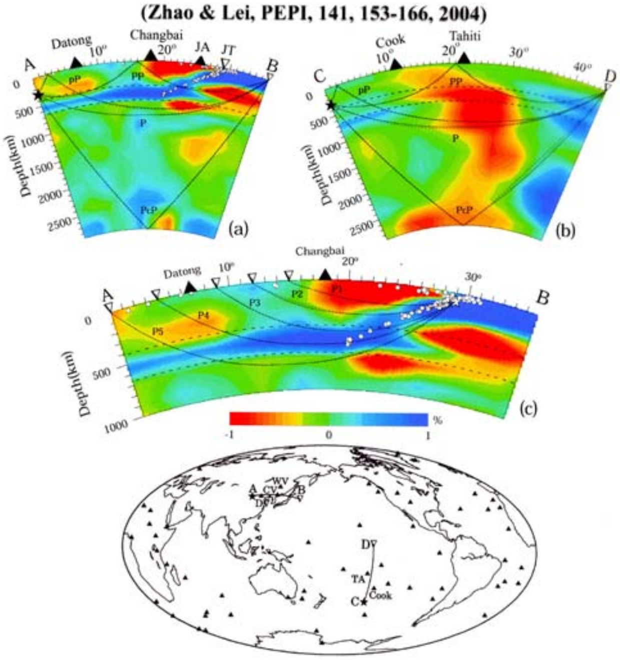


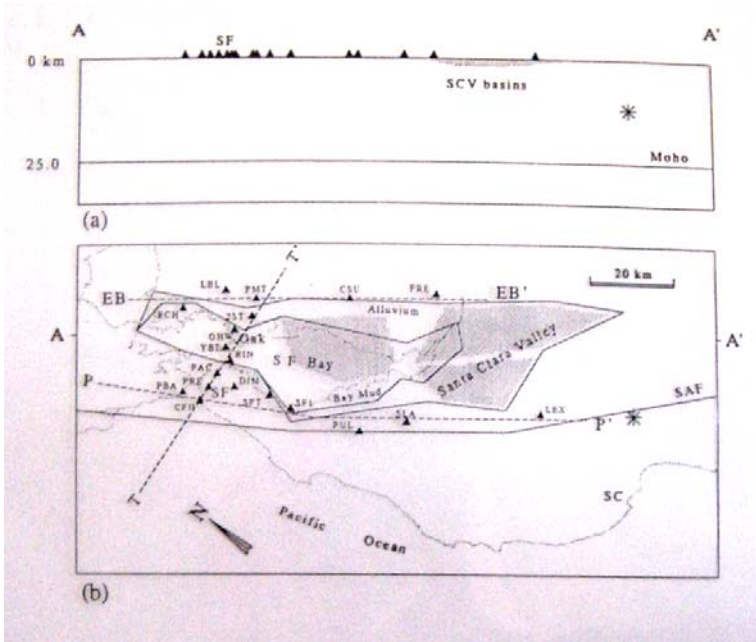
## Moho Velocity




## Seismic ray paths in a 3-D Earth model

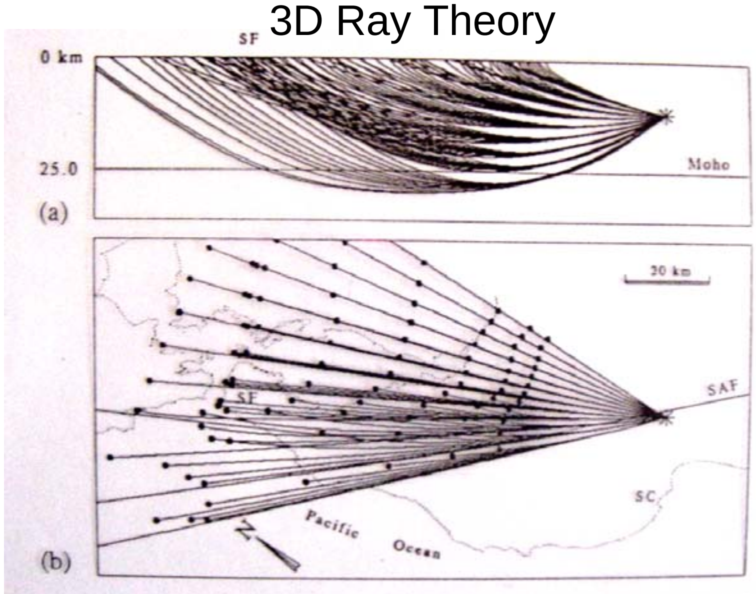
(Zhao & Lei, PEPI, 141, 153-166, 2004)

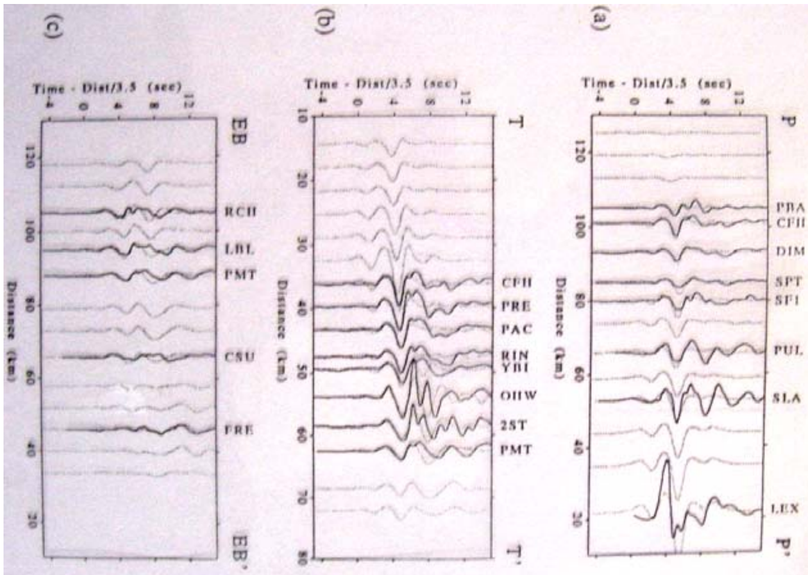





Anthony Lomax Thesis 1992

# 3D Ray Theory



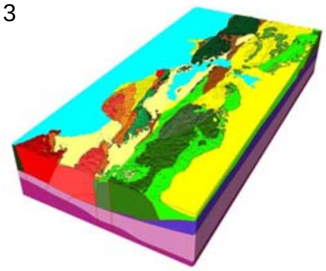



## Summary

- Classical ray theory results from a high frequency approximation to the scalar wave equation
- It is very useful for traveltime and amplitude problems for media with smoothly varying properties
- It forms a basis for body-wave tomographic inversion
- It unfortunately does not properly account for first-order discontinuities nor (easily) multiplicity of arrivals
- Full wave theory solutions are needed for study of waves in complex 3D structure with first order discontinuities such as faults and basins

# Full Wave Solutions

USGS 3D Velocity Model
Version 3



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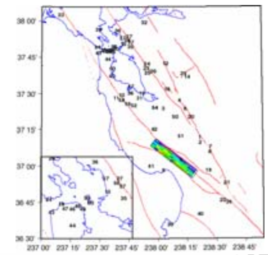
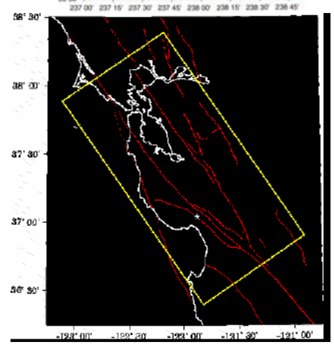


+

$$\frac{\partial \sigma_{ij}}{\partial t} = \lambda \frac{\partial v_i}{\partial x_i} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

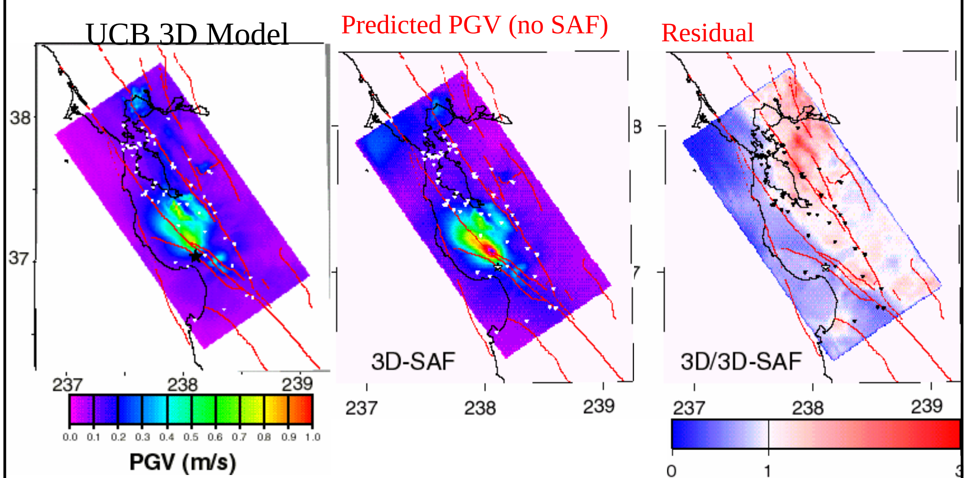
$$\rho \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i$$

=



1989 Loma Prieta Simulation
Stidham, Larsen and Dreger, 1999

# San Andreas Fault Sensitivity




Amplification (red)

# Basin Sensitivity & Amplification