Elasticity, Equation of Motion and Wave Solutions II

Consider a case in which wave particle motions are in the same direction as the wave propagation

$$\vec{u} = u(x_1)\,\hat{x}_1$$

then by substitution

$$\rho \ddot{u}\,\hat{x}_1 = (\lambda + 2\mu)\,\nabla\left(\frac{\partial u(x_1)}{\partial x_1}\right) - \mu \nabla \times \begin{vmatrix} i & j & k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ u(x_1) & 0 & 0 \end{vmatrix}$$

$$\rho \ddot{u}\,\hat{x}_1 = (\lambda + 2\mu)\,\frac{\partial^2 u}{\partial x_1^2}\,\hat{x}_1$$

$$\ddot{u}\,\hat{x}_1 = \alpha^2\,\frac{\partial^2 u}{\partial x_1^2}\,\hat{x}_1$$

Suppose a wave has particle motions that are perpendicular to the direction of propagation

$$\vec{u} = u(x_1)\,\hat{x}_2$$

then by substitution

$$\rho\ddot{u}\,\hat{x}_2 = (\lambda+2\mu)\nabla\left(\frac{\partial u(x_1)}{\partial x_2}\right) - \mu\nabla\times\begin{vmatrix} i & j & k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ 0 & u(x_1) & 0 \end{vmatrix}$$

$$= -\mu\nabla\times\left(\frac{\partial u(x_1)}{\partial x_1}\hat{x}_3\right)$$

$$= -\mu\begin{vmatrix} i & j & k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ 0 & 0 & \frac{\partial u(x_1)}{\partial x_1} \end{vmatrix}$$

$$= \mu\frac{\partial u(x_1)}{\partial x_1^2}\hat{x}_2$$

$$\ddot{u}\,\hat{x}_2 = \beta^2\frac{\partial u(x)}{\partial x_1^2}\hat{x}_2$$

Helmholtz Decomposition

$$\rho\ddot{\vec{u}} = \vec{F} + (\lambda+2\mu)\nabla(\nabla\cdot\vec{u}) - \mu\nabla\times\nabla\times\vec{u} \qquad \textbf{1}$$

$$\vec{u} = \nabla\phi + \nabla\times\vec{\Psi} \qquad \textbf{2}$$

$$\nabla\cdot\vec{\Psi} = 0$$

$$\vec{\Psi} = \langle 0,\Omega,\chi\rangle$$

ϕ, Ω, χ, are scalar displacement potential functions. Application of the above relationships to the vector wave equation results in three separated scalar wave equations for P, SV and SH waves

Substitution of 2 into 1

$$\nabla\ddot{\phi} + \nabla\times\ddot{\vec{\psi}} = \alpha^2\nabla(\nabla\cdot(\nabla\phi + \nabla\times\vec{\psi}) - \beta^2\nabla\times\nabla\times(\nabla\phi + \nabla\times\vec{\psi})$$

$$\left[\nabla\ddot{\phi} - \alpha^2\nabla(\nabla\cdot\nabla\phi)\right] + \left[\nabla\times\vec{\psi} + \beta^2\nabla\times\nabla\times\nabla\times\vec{\psi}\right] = 0$$

$$\nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times\nabla\times\vec{u}$$

$$\nabla\times(\nabla u) = 0$$

$$\nabla\cdot(\nabla\times\vec{u}) = 0$$

S-waves

$$\nabla\times\ddot{\vec{\psi}} + \beta^2\nabla\times\nabla\times\nabla\times\vec{\psi} = 0$$

$$\ddot{\vec{\psi}} + \beta^2\left[\nabla(\nabla\cdot\vec{\psi}) - \nabla^2\vec{\psi}\right] = 0$$

$$\ddot{\vec{\psi}} - \nabla^2\vec{\psi} = 0$$

$\ddot{\Omega} - \beta^2\nabla^2\Omega = 0$ SV waves

$\ddot{\chi} - \beta^2\nabla^2\chi = 0$ SH waves

$$\vec{u} = \nabla\phi + \nabla\times\vec{\Psi}$$
$$= \nabla\phi + \nabla\times\langle 0, \Omega, \chi\rangle$$

P-waves

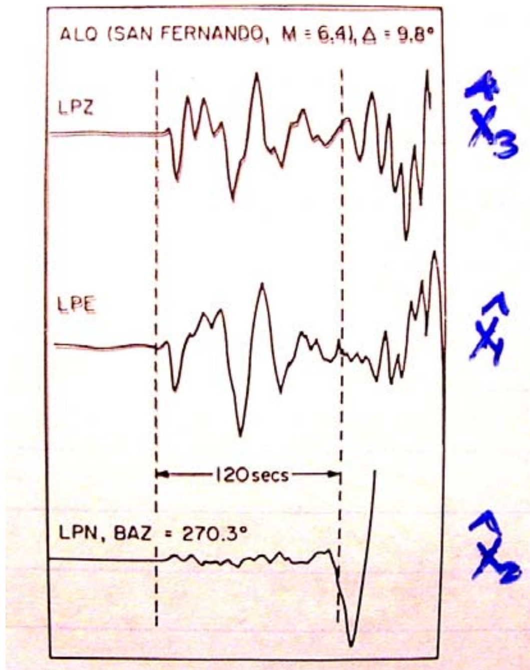
$$\nabla\ddot{\phi} - \alpha^2 \nabla(\nabla \cdot \nabla\phi) = 0$$

$$\ddot{\phi} - \alpha^2 (\nabla \cdot \nabla\phi) = 0$$

$$\ddot{\phi} - \alpha^2 \nabla^2 \phi = 0$$ P waves

$$\vec{u} = \nabla\phi + \nabla \times \vec{\Psi}$$
$$= \nabla\phi + \nabla \times \langle 0, \Omega, \chi \rangle$$

- Why go to such trouble?
  - Solutions to scalar wave equations are simpler.
- What has been assumed?
  - Homogeneous media leading to 1D boundary value problems
- How is vector motion found?
  - Solve each scalar wave equation separately and then combine using Helmholtz equation



Actual records too seem to be organized in this way

Different Coordinate Systems

$$\ddot{\phi} - \alpha^2 \nabla^2 \phi = 0$$

Cartesian Geometry

$$\phi(\vec{x},t) = f(t - \vec{x}/\alpha) + g(t + \vec{x}/\alpha)$$

General solutions
Specific solutions are harmonic functions
This geometry is used for local to regional scale 3D wave propagation problems

$$\ddot{\phi} - \alpha^2 \left[\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \right] = 0$$

Cylindrical Geometry

$$\phi(\vec{x},t) = \frac{f(t - R/\alpha)}{R}$$

General Solution
Specific solutions are Bessel functions

It is useful to think of f(t-R/α) as f(t)*δ(t-R/α)

This geometry is often used for 1D wave propagation problems

Different Coordinate Systems

$$\ddot{\phi} - \alpha^2\left[\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)\right)\right] = 0$$

Spherical Geometry

$$\phi(\vec{x},t) = \frac{f(t - r/\alpha)}{r}$$

General Solution
Specific solutions are Legendre polynomials

This coordinate system is used for whole Earth scale problems

Solutions to Scalar Wave Equations

$$\ddot{\phi} - \alpha^2\nabla^2\phi = 0$$

$$\phi(x_1,t) = X(x_1)T(t)$$

$$X(x_1)\frac{d^2T}{dt^2} = \alpha^2 T(t)\frac{d^2X}{dx_1^2} = -\omega^2$$

$$\frac{d^2T}{dt^2} + \omega^2 T = 0$$

$$\frac{d^2X}{dx_1^2} + \frac{\omega^2}{\alpha^2}X = 0$$

$$T(t) = be^{\pm i\omega t}$$

$$X(x_1) = ce^{\pm ikx} \qquad k = \frac{\omega}{\alpha}$$

$$\phi(x_1,t) = Ae^{\pm i(\omega t + kx_1)}$$

$$\phi(\vec{x},t) = Ae^{i\left(\pm\omega t \pm \vec{k}\cdot\vec{x} + \varepsilon\right)}$$

$$\vec{k} = k_1\hat{x}_1 + k_2\hat{x}_2 + k_3\hat{x}_3 = \langle k_1, k_2, k_3 \rangle$$

$$\left\|\vec{k}\right\| = \sqrt{k_1^2 + k_2^2 + k_3^2} = \frac{\omega}{\alpha}$$

Green's Function Solution

$$\ddot{\phi} - \alpha^2\nabla^2\phi = -4\pi\alpha^2\delta(x)\delta(y)\delta(z)\delta(t)$$

$$\left[\frac{s^2}{\alpha^2} + \left(k_1^2 + k_2^2 + k_3^2\right)\right]\tilde{\phi} = -4\pi$$

$$\phi(\bar{x},s) = \frac{-4\pi}{8\pi^3}\iiint \frac{e^{ik_x x}\cdot e^{ik_y y}\cdot e^{ik_z z}}{\left[\frac{s^2}{\alpha^2} + \left(k_1^2 + k_2^2 + k_3^2\right)\right]} dk_x dk_y dk_z$$

$$\phi(r,z,s) = \int_0^\infty J_0(k_r r) e^{-\gamma z} \frac{k_r}{\gamma} = \frac{e^{-Rs/\alpha}}{R}$$

$$\int_0^\infty \delta\left(t - R/\alpha\right) \frac{1}{R} e^{-st} dt = \int_0^\infty \delta(\tau) \frac{1}{R} e^{-s\left(\tau + R/\alpha\right)} d\tau = \frac{e^{-sR/\alpha}}{R}$$

Behavior of Plane Wave Solutions

$$\ddot{\phi} - \alpha^2 \nabla^2 \phi = 0$$

$$\phi(\vec{x},t) = Ae^{i\left(\pm\omega t \pm \vec{k}\cdot\vec{x} + \varepsilon\right)}$$

Solutions found by substitution, Fourier Transforms, separation of variables

Substitution into scalar wave equation yields dispersion relation

$$\omega^2 - \alpha^2\left(k_1^2 + k_2^2 + k_3^2\right) = 0$$

$$k_1^2 + k_2^2 + k_3^2 = \frac{\omega^2}{\alpha^2}$$

$$\nabla\phi = \vec{k}\phi$$

$$\vec{k} = \left\|\vec{k}\right\|\hat{k} = \frac{\omega}{\alpha}\hat{k}$$

Vector displacement motions are obtained from the Helmholtz equation

$$\vec{u} = \nabla\phi + \nabla\times\vec{\Psi}$$
$$= \nabla\phi + \nabla\times\langle 0, \Omega, \chi\rangle$$

A constant phase plane wave has wave fronts defined by

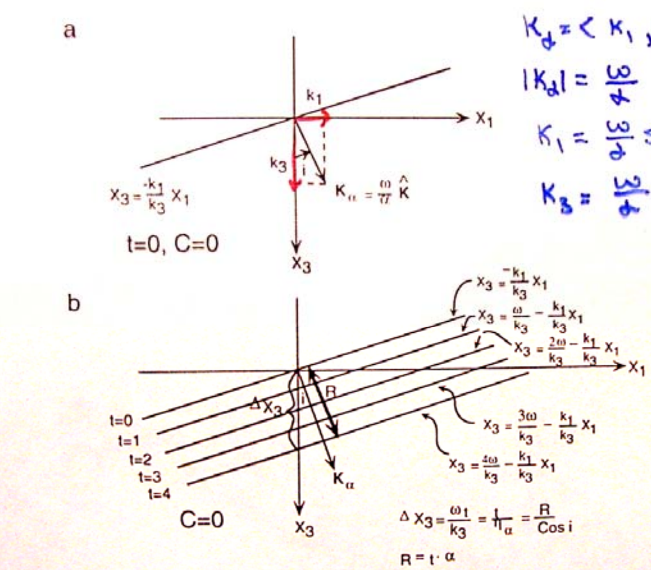
$$\pm\omega t \pm K_1 x_1 \pm K_2 x_2 \pm K_3 x_3 = C$$

* sign determines the direction of propagation

e.g. in the $\hat{x}_1 \hat{x}_3$ plane

$$\omega t - K_1 x_1 - K_3 x_3 = C \qquad t = \frac{[C + K_1 x_1 + K_3 x_3]}{\omega}$$

$$x_3 = \frac{\omega t}{k_3} - \frac{k_1}{k_3} x_1$$



$$K_\alpha = \langle K_1, 0, K_3 \rangle$$

$$|K_\alpha| = \frac{\omega}{\alpha}$$

$$K_1 = \frac{\omega}{\alpha} \sin i = \omega p$$

$$K_3 = \frac{\omega}{\alpha} \cos i = \eta \omega$$

FIGURE 2.11 (a) The projection of the wavefront defined by $t = 0$, $C = 0$ in the $x_1 x_3$ plane and the associated wavenumber vector $\mathbf{k}_\alpha$. (b) Variation of the position of a wavefront of constant phase ($C = 0$) for increasing time, t. The distance that the wavefront moves after time t is $R = \alpha t$.

Plane Wave Displacement Motions

P-waves:

$$u_p = \nabla \phi$$

* Note that the gradient of a scalar field also gives the direction that the wave is traveling

For the $x_1 x_3$ problem

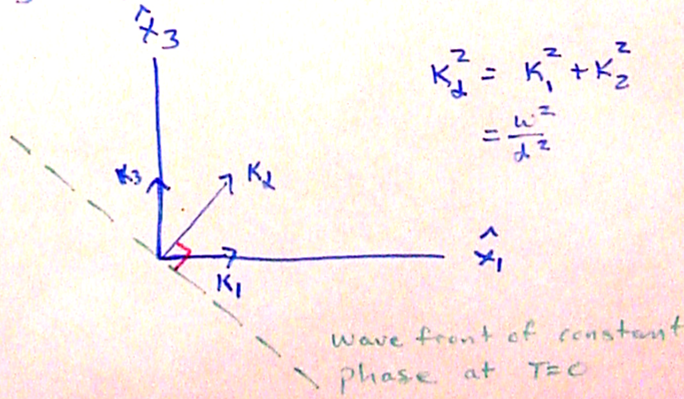
$$\phi(\vec{x}, t) = e^{i(\omega t + k_1 x_1 + k_3 x_3)}$$

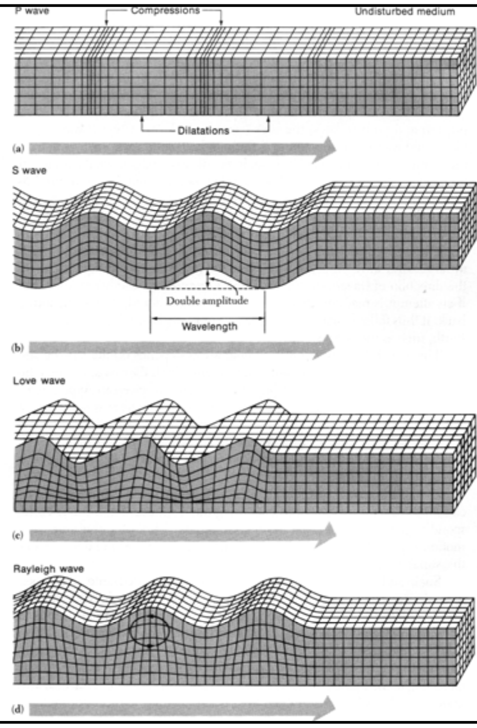
$$u_p(\vec{x}, t) = (i k_1 \hat{x}_1 + i k_3 \hat{x}_3)\, e^{i(\omega t + k_1 x_1 + k_3 x_3)}$$

The ratio of horizontal to vertical motions is then

$$\frac{u_1}{u_3} = \frac{k_1}{k_3} \quad \Rightarrow \quad u_3 = \frac{k_3}{k_1} u_1$$

$$k_\alpha^2 = k_1^2 + k_2^2 = \frac{\omega^2}{\alpha^2}$$





SV-waves

$$\Omega(\vec{x},t) = A e^{i(\omega t + k_1 x_1 + k_3 x_3)}$$

direction $\nabla\Omega = \langle i k_1, 0, i k_3 \rangle A e^{i(\omega t + k_1 x_1 + k_3 x_3)}$

Particle motions are:

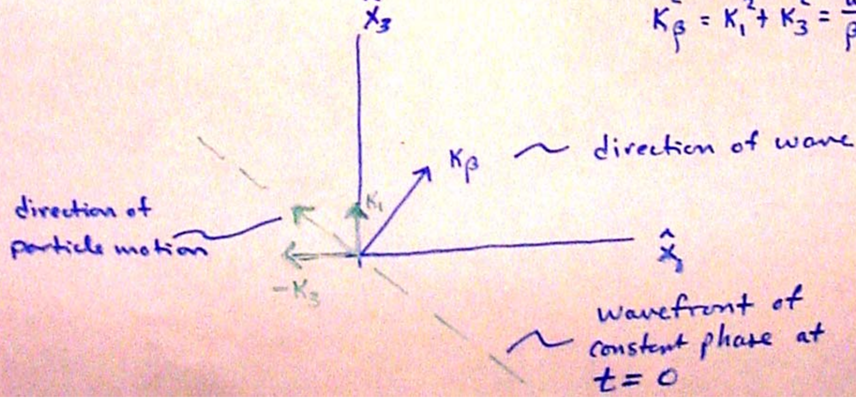
$$u_{sv}(\vec{x},t) = \nabla \times \langle 0, \Omega, 0 \rangle$$

$$= \left\langle -\frac{d\Omega}{dx_3}, 0, \frac{d\Omega}{dx_1} \right\rangle$$

$$= A \langle -i k_3, 0, i k_1 \rangle e^{i(\omega t + k_1 x_1 + k_3 x_3)}$$

$$\frac{U_1}{U_3} = \frac{-K_3}{K_1} \Rightarrow U_3 = -K_1/K_3\, U_1$$

$$K_\beta^2 = K_1^2 + K_3^2 = \frac{\omega^2}{\beta^2}$$



SH waves

$$\chi = A e^{i(\omega t + K_1 x_1 + K_3 x_3)}$$

direction $\propto \nabla \chi$

Particle motions, $U_{SH} = \nabla \times \langle 0, 0, \chi \rangle$

$$= \langle \frac{d\chi}{dx_2}, -\frac{d\chi}{dx_1}, 0 \rangle$$

$$= A \langle 0, -iK_1, 0 \rangle e^{i(\omega t + K_1 x_1 + K_3 x_3)}$$

$$= A K_1 \sin(\omega t + K_1 x_1 + K_3 x_3)$$

since $e^{i\theta} = \cos\theta + i \sin\theta$

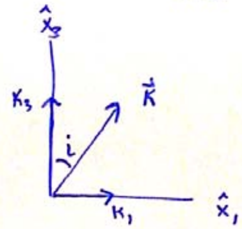
Another very useful notation

Given:

$$\phi(\vec{x},t) = A e^{i(\omega t - K_1 x_1 - K_3 x_3)}$$

$$= A e^{i\omega(t - \frac{K_1}{\omega} x_1 - \frac{K_3}{\omega} x_3)}$$

$$= A e^{i\omega(t - p x_1 - \eta x_3)}$$



$K = \langle K_1, 0, K_3 \rangle$

$|K| = \omega/\alpha$

$K_1 = |K| \sin i = \frac{\omega}{\alpha} \sin i$

$K_3 = |K| \cos i = \frac{\omega}{\alpha} \cos i$

$\frac{K_1}{\omega} = \frac{\sin i}{\alpha} = p$

$\frac{K_3}{\omega} = \frac{\cos i}{\alpha} = \eta$

For an initial phase of zero

$$t - px - \eta z = 0$$

** $t = px + \eta z$

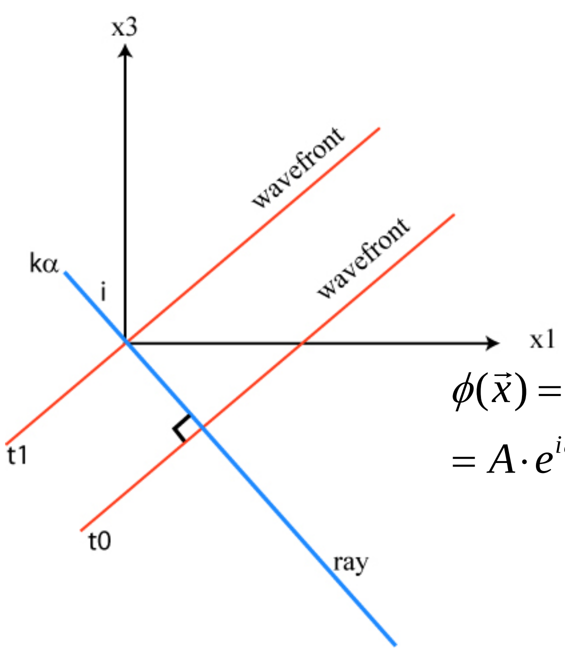
arrival time — t

horizontal slowness $\cdot x$ = horizontal traveltime

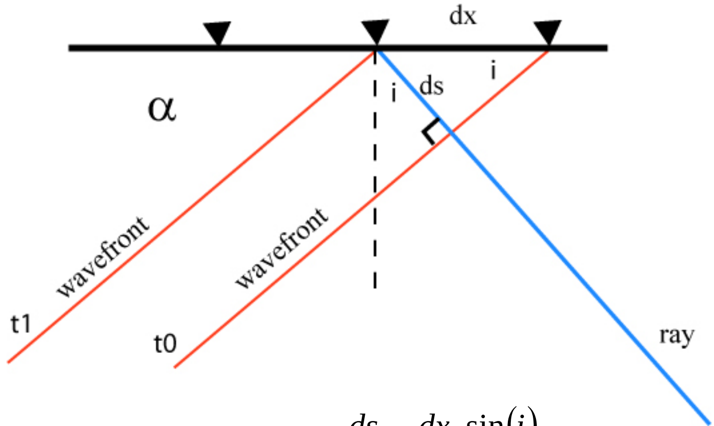
vertical slowness $\cdot z$ = vertical traveltime

Exercise

- How can we define an experiment making use of plane wave theory to determine the velocity of the Earth's surface?
  - Assume a teleseismic (greater than 3000 km) plane wave arrival.
  - What can we measure?
  - How can we determine α?

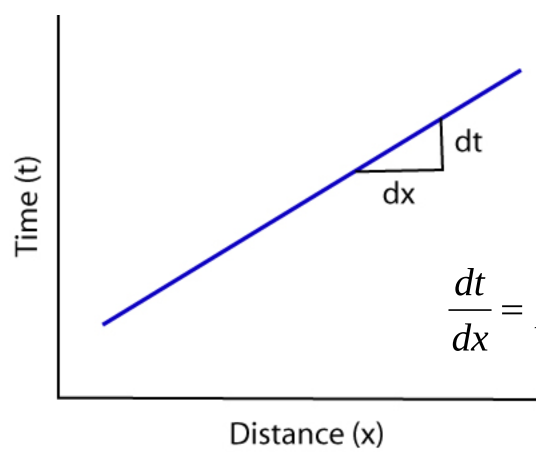


$$\phi(\vec{x}) = A \cdot e^{i\omega\left(t + k_1 x_1 - k_3 x_3\right)}$$
$$= A \cdot e^{i\omega\left(t + p x_1 - \eta x_3\right)}$$



$$dt = \frac{ds}{\alpha} = \frac{dx \cdot \sin(i)}{\alpha}$$

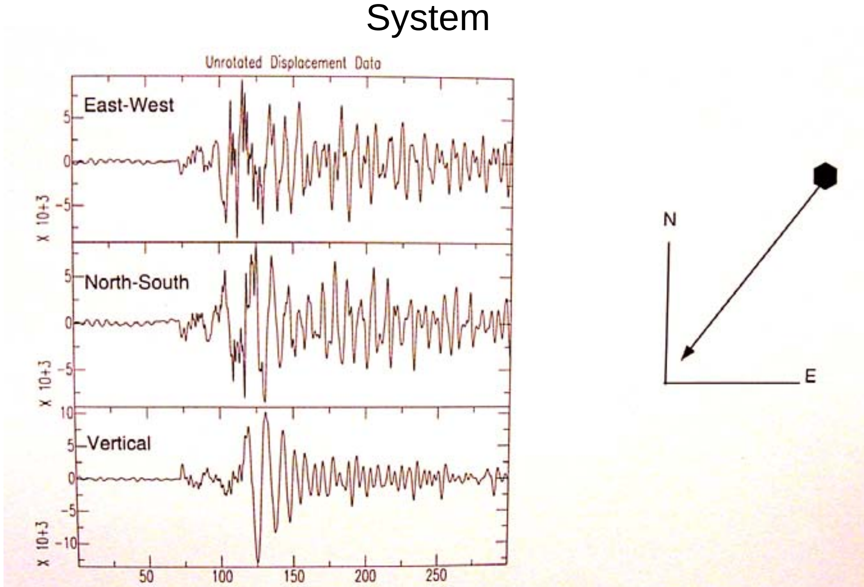
$$\frac{dt}{dx} = \frac{\sin(i)}{\alpha} = p$$



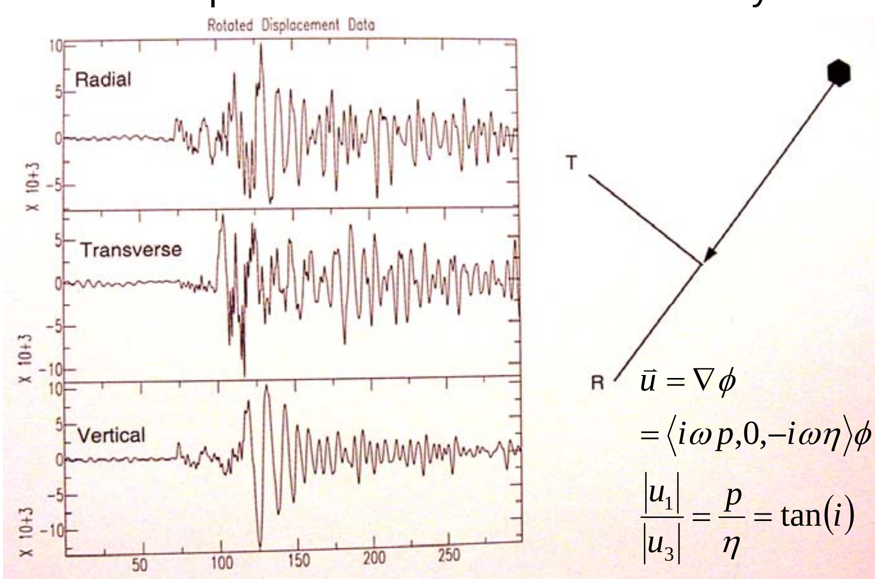
$$\frac{dt}{dx} = p = \frac{\sin(i)}{\alpha}$$

How can we determine *i* ?

Three-Components in Geographic Coordinate System



Three-Components in Rotated Coordinate System



$$\vec{u} = \nabla \phi$$

$$= \langle i\omega p, 0, -i\omega\eta \rangle \phi$$

$$\frac{|u_1|}{|u_3|} = \frac{p}{\eta} = \tan(i)$$