Exercise 1 Show $\Phi(r,z,t) = \dfrac{f(t-R/v)}{R}$ (1)

where $R = [r^2+z^2]^{1/2}$ is a solution of

the cylindrical wave equation

(2) $$\frac{\partial^2\Phi}{\partial t^2} - v^2\left[\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}\right] = 0$$

Verify by substitution

Take Laplace transform to simplify (1) & (2)

$$\tilde{\Phi}(r,z,s) = \frac{f(s)\,e^{-Rs/v}}{R} \quad \&$$

$$s^2\tilde{\Phi} - v^2\left[\frac{\partial^2\tilde{\Phi}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{\Phi}}{\partial r} + \frac{\partial^2\tilde{\Phi}}{\partial z^2}\right] = 0$$

note: azimuthal term of (2) vanishes because of symetry of spherical source

$$\frac{\partial\tilde{\Phi}}{\partial r} = \left[-\frac{1}{2}\frac{2r}{R^3}e^{-Rs/v} - \frac{1}{2}\frac{2rs}{vR^2}e^{-Rs/v}\right]f(s)$$

$$= \left[-\frac{r}{R^3}e^{-Rs/v} - \frac{rs}{vR^2}e^{-Rs/v}\right]f(s)$$

$$\boxed{\frac{1}{r}\frac{\partial\tilde{\Phi}}{\partial r} = \left[-\frac{e^{-Rs/v}}{R^3} - \frac{s}{vR^2}e^{-Rs/v}\right]f(s)}$$

$$\frac{d^2\hat{\Phi}}{dr^2} = \Big[-\frac{1}{R^3} e^{-RS/V} + 3\frac{r^2}{R^5} e^{-RS/V} + \frac{r^2 s}{R^4 V} e^{-RS/V} - \frac{s}{VR^2} e^{-RSV} + \frac{2r^2 s}{VR^4} e^{-RS/V}$$

$$+ \frac{r^2 s^2}{V^2 R^3} e^{-RS/V} \Big] f(s)$$

$$= -\frac{1}{R^3} e^{-RS/V} + 3\frac{r^2}{R^5} e^{-RS/V} + 3\frac{r^2 s}{VR^4} e^{-RS/V} - \frac{s}{VR^2} e^{-RS/V} + \frac{r^2 s^2}{V^2 R^3} e^{-\frac{RS}{V}} \Big] f(s)$$

$$\frac{d^2\hat{\Phi}}{dz^2} = -\frac{1}{R^3} e^{-RS/V} + 3\frac{z^2}{R^5} e^{-RS/V} + 3\frac{z^2 s}{VR^4} e^{-RS/V} - \frac{s}{VR^2} e^{-RS/V} + \frac{z^2 s^2}{V^2 R^3} e^{-\frac{RS}{V}} \Big] f(s)$$

$$\boxed{\frac{d^2\hat{\Phi}}{dr^2} + \frac{d^2\hat{\Phi}}{dz^2} = -{}^2/R^3 e^{-RS/V} + 3\frac{1}{R^3} e^{-RS/V} + 3\frac{s}{VR^2} e^{-RS/V} - \frac{2s}{VR^2} e^{-RS/V} + \frac{s^2}{V^2 R} e^{-\frac{RS}{V}} \Big] f(s)}$$

$$\frac{d^2\hat{\Phi}}{dr^2} + \frac{1}{r}\frac{d\hat{\Phi}}{dr} + \frac{d^2\hat{\Phi}}{dz^2} = -3\frac{1}{R^3} e^{-RS/V} + 3\frac{1}{R^3} e^{-RS/V} + 3\frac{s}{VR^2} e^{-RS/V} - 3\frac{s}{VR^2} e^{-RS/V} + \frac{s^2}{V^2 R} e^{-\frac{RS}{V}} \Big] f(s)$$

$$= \frac{s^2}{V^2 R} e^{-RS/V} f(s)$$

$\therefore$ from ②

$$s^2 \frac{f(s) e^{-RS/V}}{R} - V^2 \Big[\frac{s^2}{V^2 R} e^{-RS/V} f(s) \Big] = 0 \quad \checkmark$$

Solution Set

Exercise 2

$$\phi = 7e^{2i\left(\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 - 6t\right)}$$

$$\vec{\psi} = \langle \sqrt{3}, -1, 6 \rangle e^{3i\left(\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 - 4t\right)}$$

A) WAVE PROPAGATION DIRECTION

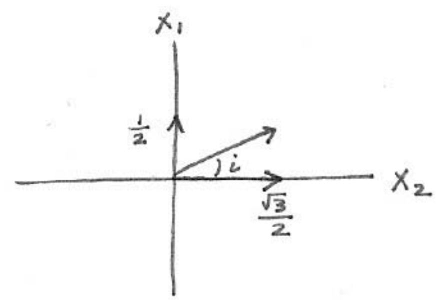
For both P and S waves:

Propagation direction for unit vector

$$\langle N, E, Z \rangle = \underbrace{\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle}_{\text{AMPLITUDE}}$$

$$\tan i = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$i = 30°$ N60E



B) P-WAVE PARTICLE MOTIONS

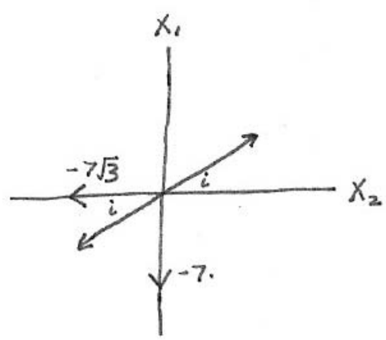
$$\vec{U} = \nabla\phi = \left\langle 7 \cdot 2i \cdot \frac{1}{2}, 7 \cdot 2i \cdot \frac{\sqrt{3}}{2}, 0 \right\rangle e^{2i\left(\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 - 6t\right)}$$

Take real part knowing $e^{ix} = \cos x + i \sin x$

$$= \underbrace{\langle -7, -7\sqrt{3}, 0 \rangle}_{\text{AMPLITUDE}} \sin(x_1 + \sqrt{3}x_2 - 12t)$$

$$\tan i = \frac{-7}{-7\sqrt{3}}$$

$i = 30°$



$\Rightarrow$ Azimuth

Exercise 2

c) S-WAVE PARTICLE MOTION

$$\vec{U} = \nabla \times \vec{\Psi} = \langle 9i\sqrt{3}, -9i, -6i \rangle e^{3i(\frac{1}{2}x_1 + \frac{3\sqrt{3}}{2}x_2 - 12t)}$$

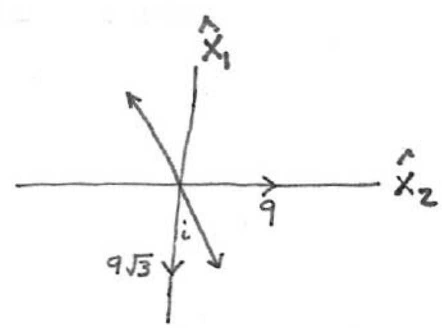
Take only the real part

$$= \underbrace{\langle -9\sqrt{3}, 9, 6 \rangle}_{\text{AMPLITUDE}} \sin(\tfrac{3}{2}x_1 + \tfrac{3\sqrt{3}}{2}x_2 - 12t)$$

Since $A_3 \neq 0$, this vector is in 3 dimensions, unlike P-wave, and we will solve for plunge as well.

$$\tan i = \frac{9}{-9\sqrt{3}}$$

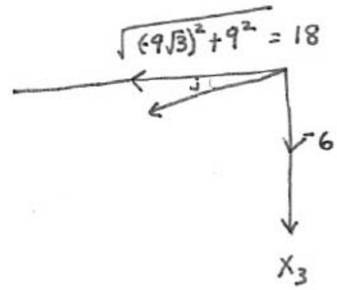
$$i = -30°$$



Azimuth

$$\tan j = \frac{-6}{18}$$

$$j = -18.4$$



Plunge

Exercise 3

A) Find $\phi(\vec{x},t) = Ae^{i(\ t - K_1X_1 - K_2X_2 - K_3X_3)}$

(Although any variation of $Ae^{\pm i(\omega t \pm K_1X_1 \pm K_2X_2 \pm K_3X_3)}$ is okay)

WE KNOW

$$\vec{U}_p = \left\langle \frac{4}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{4}{\sqrt{3}} \right\rangle \qquad \omega = 4 \text{ rad/s} \qquad \alpha = 6 \text{ km/s}$$

$$\vec{K} = \langle K_1, K_2, K_3 \rangle$$

$$\|\vec{K}\|^2 = K_1^2 + K_2^2 + K_3^2 = \frac{\omega^2}{\alpha^2} = \frac{4}{9}$$

SET $\vec{U}_p$ and $\nabla\phi$ equal to eachother

$$\vec{U}_p = \nabla\phi$$

$$\left\langle \frac{4}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{4}{\sqrt{3}} \right\rangle = \langle -AK_1, -AK_2, -AK_3 \rangle$$

$$\left\langle \frac{4}{A3\sqrt{2}}, \frac{4}{A3\sqrt{2}}, \frac{4}{A\sqrt{3}} \right\rangle = -\vec{K}$$

Then

$$\|\vec{K}\|^2 = \frac{4}{9} = \left(\frac{4}{A3\sqrt{2}}\right)^2 + \left(\frac{4}{A3\sqrt{2}}\right)^2 + \left(\frac{4}{A\sqrt{3}}\right)^2$$

$$A^2 = 16$$

$$A = 4$$

AND

$$\vec{K} = \left\langle \frac{-1}{3\sqrt{2}}, \frac{-1}{3\sqrt{2}}, \frac{-1}{\sqrt{3}} \right\rangle$$

$$\phi(\vec{x},t) = 4e^{i(4t + \frac{1}{3\sqrt{2}}X_1 + \frac{1}{3\sqrt{2}}X_2 + \frac{1}{\sqrt{3}}X_3)}$$

Exercise 3

B) FIND $\vec{\Psi}(\vec{x},t) = \langle K_1, K_2, K_3 \rangle e^{i(\omega t \mp x_1 k_1 \mp x_2 k_2 \mp x_3 k_3)}$

We Know

$\vec{U}_s = \langle -\sqrt{3}, -\sqrt{3}, \sqrt{2} \rangle \quad \omega = 4 \text{ rad/s} \quad \beta = 4 \text{ km/s}$

SIMPLIFY THE PROBLEM

This is an SV wave so rotate it into a new $x_1' x_3'$ coordinate frame

$\vec{U}_s' = \langle \sqrt{6}, 0, \sqrt{2} \rangle$

ALSO ASSUME

$\vec{\Psi} = \langle 0, \Omega, 0 \rangle$

since then $\nabla \times \Psi$ will only have x_1 and x_3 components - SV like $\vec{U}_s'$

THEN WE CAN FIND

$\vec{U}_s' = \nabla \times (0, \Omega, 0)$

$\vec{U}_s' = \langle -\frac{\partial \Omega}{\partial x_3}, 0, \frac{\partial \Omega}{\partial x_1} \rangle$

WHERE

$\Omega(\vec{x},t) = A e^{i(4t - k_1' x_1' - k_3' x_3')}$

SETING $\vec{U}_s'$ AND $\nabla \times \Psi$ equal

$\langle \sqrt{6}, 0, \sqrt{2} \rangle = \langle +A k_3', 0, -A k_1' \rangle$

$\|\vec{K}\|^2 = \frac{6}{A^2} + \frac{2}{A^2} = \frac{16}{16}$

$A = 2\sqrt{2}$

Exersize 3

FIND $\vec{K}'$

$$\vec{U}_s' = \nabla \times \Psi$$

$$\langle \sqrt{6}, 0, \sqrt{2} \rangle = \langle 2\sqrt{2}\, K_3', 0, -2\sqrt{2}\, K_1' \rangle$$

$$\langle \tfrac{\sqrt{3}}{2}, 0, -\tfrac{1}{2} \rangle = \langle K_3', 0, K_1' \rangle$$

Return to Assumed Ω to Find Correct Sign

$$\Omega(\vec{x}, t) = 2\sqrt{2}\, e^{i(4t + \frac{1}{2}x_1' - \frac{\sqrt{3}}{2}x_3')}$$

$$\vec{K}' = \langle \tfrac{1}{2}, 0, -\tfrac{\sqrt{3}}{2} \rangle$$

ROTATE BACK

$$\vec{K} = \langle -\tfrac{\sqrt{2}}{4}, -\tfrac{\sqrt{2}}{4}, -\tfrac{\sqrt{3}}{2} \rangle = \langle \tfrac{1}{2}\cos\tfrac{5\pi}{4}, \tfrac{1}{2}\sin\tfrac{5\pi}{4}, -\tfrac{\sqrt{3}}{2} \rangle$$

FIND $\vec{B}$

$$\vec{\Psi} = \langle B_1, B_2, B_3 \rangle e^{i(4t - \frac{\sqrt{2}}{4}x_1 - \frac{\sqrt{2}}{4}x_2 - \frac{\sqrt{3}}{2}x_3)}$$

KNOWING

$$\nabla \times \vec{\Psi} = \langle -\sqrt{3}, -\sqrt{3}, \sqrt{2} \rangle$$

$$\nabla \cdot \vec{\Psi} = 0$$

LEADS TO:

$$0 - B_2\left(-\tfrac{\sqrt{3}}{2}\right) + B_3\left(-\tfrac{\sqrt{2}}{4}\right) = -\sqrt{3}$$

$$B_1\left(\tfrac{-\sqrt{3}}{2}\right) + 0 - B_3\left(\tfrac{-\sqrt{2}}{4}\right) = -\sqrt{3}$$

$$-B_1\left(\tfrac{-\sqrt{2}}{4}\right) + B_2\left(\tfrac{-\sqrt{2}}{4}\right) + 0 = \sqrt{2}$$

$$B_1\left(\tfrac{-\sqrt{2}}{4}\right) + B_2\left(\tfrac{-\sqrt{2}}{4}\right) + B_3\left(\tfrac{-\sqrt{3}}{2}\right) = 0$$

SEE MATHCAD SHEET FOR SOLUTION

$$A := \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & \frac{-\sqrt{2}}{4} \\ \frac{-\sqrt{3}}{2} & 0 & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} & 0 \\ \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{3}}{2} \end{bmatrix} \qquad D := \begin{bmatrix} -\sqrt{3} \\ -\sqrt{3} \\ \sqrt{2} \\ 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -0.866 & 0.354 & -0.354 \\ 0.866 & 0 & -0.354 & -0.354 \\ -0.354 & 0.354 & 0 & -0.866 \end{bmatrix} \qquad A^T \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M := \left(A^T \cdot A\right)^{-1} \cdot A^T \cdot D$$

$$M = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

HENCE SOLUTION IS

$$\vec{\psi} = \langle 2, -2, 0 \rangle e^{i\left(4t - \frac{\sqrt{2}}{4}x_1 - \frac{\sqrt{2}}{4}x_2 - \frac{\sqrt{3}}{2}x_3\right)}$$

CHECK

$$\nabla \cdot \vec{\psi} = \left\langle -(-2)\left(-\frac{\sqrt{3}}{2}\right),\ (2)\left(-\frac{\sqrt{3}}{2}\right),\ -(2)\left(-\frac{\sqrt{2}}{4}\right) + (-2)\left(-\frac{\sqrt{2}}{4}\right) \right\rangle$$

$$= \langle -\sqrt{3},\ -\sqrt{3},\ \sqrt{2} \rangle$$